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VARIATIONAL APPROACH BY MEANS OF ADJOINT SYSTEMS TO STRUCTURAL OPTIMIZATION AND SENSITIVITY ANALYSIS—II

STRUCTURE SHAPE VARIATION

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Abstract—For a linear elastic structure, the first variation of an arbitrary stress, strain and displacement functionals corresponding to variation of shape of external boundaries or interfaces is derived by using the solutions for primary and adjoint systems. The application to optimal design is next presented and the relevant optimality conditions are derived from general expressions. The path-independent integrals used in fracture mechanics are rederived as a particular case of general expressions.

1. INTRODUCTION

The present paper constitutes the continuation of previous work (Part I, [1]) on variational approach to sensitivity analysis with respect to design functions varying within a specified domain.The first variations of any stress, strain or displacement functionals were explicitly expressed in terms of variations of design functions. Now, a more difficult problem will be considered when external boundaries of the structure or interfaces are allowed to vary. The respective variations of the considered functionals will be expressed in terms of the variation of a transformation field specifying the shape modification. In particular, the variation of the potential and complementary energies will be considered. When the transformation field corresponds to translation or rotation of the boundary, the potential energy variation is identical to that derived previously by Eshelby[8], Knowles and Sternberg[9], Budiansky and Rice[10] and Bui[11]. The application of the derived expressions to optimal design problems will next be considered. In Section 2, the virtual displacement and stress equations will be derived, whereas in Section 3 the variation of the potential and complementary energies will be considered. In Section 4, the variation of an arbitrary stress, strain and displacement functionals will be discussed and in Section 5 the stationarity conditions for some optimal shape design problems will be derived. However, the significance of the obtained results is much broader as they can be applied in fracture mechanics or in studying growth of biological structures or metallurgical transformations. Some simple illustrative examples are presented in Section 6. The results previously obtained in [2–5] for optimal shape design are incorporated in a much broader context. A variational approach to optimal shape design was also discussed in [6, 7] and in books [13, 14].

2. VIRTUAL DISPLACEMENT AND STRESS EQUATIONS FOR STRUCTURES WITH VARYING LOADED, FREE, SUPPORTED AND INTERNAL BOUNDARIES

Consider now an elastic body B occupying the domain V with the boundary S. The surface tractions $\mathbf{T}^0 = \boldsymbol{\sigma} \cdot \mathbf{n}$ are specified on S_T, displacements $\mathbf{u} = \mathbf{u}^0$ on S_u. Under applied loads, the body passes from its initial configuration C to a deformed configuration C_d specified by the displacement field $\mathbf{u}$, $\mathbf{x}^d = \mathbf{x} + \mathbf{u}$. Besides a *deformation process* $C \to C_d$ consider a *transformation process* $C \to C_t$, $\mathbf{x}^t = \mathbf{x} + \boldsymbol{\varphi}$ with the imposed transformation field $\boldsymbol{\varphi}(\mathbf{x})$ specified within V, Fig. 1. Obviously, this transformation field modifies shape of external boundaries or internal interfaces between different materials. The major question

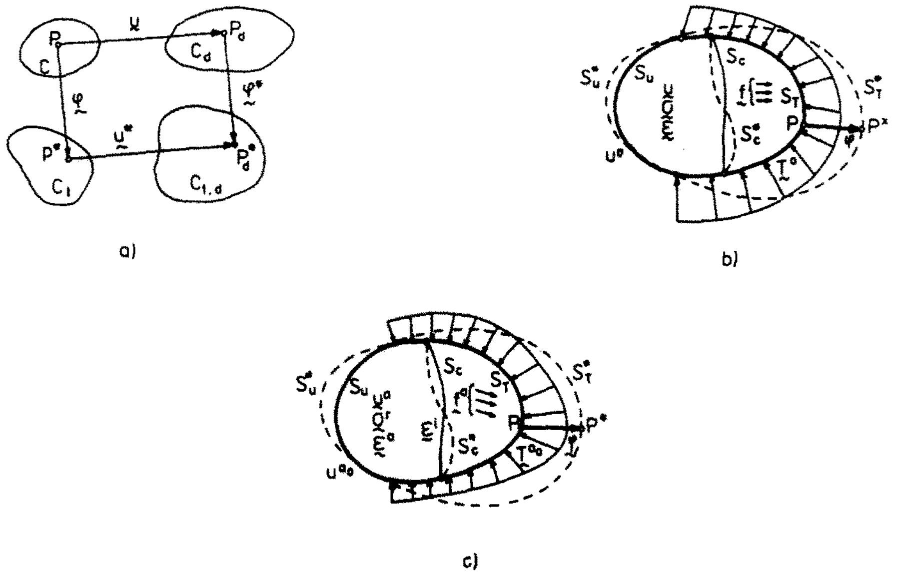

Fig. 1. (a) Transformation and deformation process of the body, (b) Primary structure of varying shape, (c) Adjoint structure for stress functional.

can now be posed as how stress, strain, displacement or some global functionals are modified due to transformation of a structure. In analysing this problem, an important constraint is imposed on the transformation field namely, the variation of a particular boundary portion, say S_T or S_u, is assumed as not affecting the remaining boundary portions. Thus, the variation of each boundary part or interface can be treated separately. This restriction limits the class of shape variations since in actuality the shape modification may occur with simultaneous variation of all portions of the boundary with interaction effects occuring on the lines separating these portions. Such *coupled boundary variation* will be treated in a subsequent paper and here only *non-coupled variations* of particular portions will be considered. It is further assumed that $\varphi(\mathbf{x})$ is a continuous and differentiable field.

The analysis will be confined to small displacement and strain theory. Assume stress $\sigma(\mathbf{x})$, strain $\epsilon(\mathbf{x})$ and displacement $\mathbf{u}(\mathbf{x})$ within the body (referred to a Cartesian reference frame) to satisfy equilibrium, compatibility and boundary conditions. Let the reference configuration correspond to a given transformation field $\varphi(\mathbf{x})$. Consider next the infinitesimal transformation $\delta\varphi(\mathbf{x})$ of the structure and the associated variations $\delta\sigma(\mathbf{x})$, $\delta\epsilon(\mathbf{x})$ and $\delta\mathbf{u}(\mathbf{x})$. In this Section the modified forms of virtual displacement and stress equations will be derived and next applied in derivation of the functional variation.

2.1 *Virtual displacement equation*

Consider a simultaneous variation of the displacement and transformation fields. If $\mathbf{x}^*$ denotes the position of a point P, initially placed at $\mathbf{x}$, after infinitesimal variation of φ, we can write, Fig. 2(a),

$$u_i^*(\mathbf{x}^*) = u_i(\mathbf{x}) + \delta u_i(\mathbf{x}), \tag{1}$$

$$P \to P^*: \quad x_i^* = x_i + \delta\varphi_i, \tag{2}$$

where $\delta\varphi$ is a differentiable field. From (1) and (2), it follows that

$$\delta u_i = u_i^*(\mathbf{x}^*) - u_i(\mathbf{x}) = \delta\bar{u}_i + u_{i,k}\delta\varphi_k, \tag{3}$$

where comma preceding an index denotes partial differentiation. Here $\delta\bar{u}_i = u_i^*(\mathbf{x}) - u_i(\mathbf{x})$ denotes the displacement variation for a fixed configuration of the body. Assume first that the boundary portion S_u is not altered and then we have $\delta u_i = \delta\bar{u}_i = 0$ on S_u.

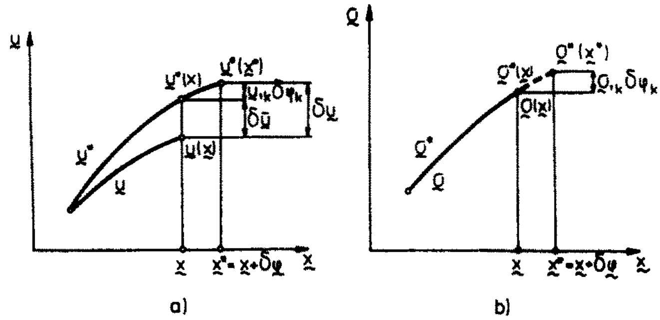

Fig. 2. Variation of displacement (a) and stress (b) fields due to boundary shape variation.

The variation of strain is expressed similarly as follows

$$\delta\epsilon_{ij} = \delta\bar{\epsilon}_{ij} + \epsilon_{ij,k}\delta\varphi_k, \quad \epsilon^*_{ij}(\mathbf{x}^*) = \epsilon_{ij}(\mathbf{x}) + \delta\epsilon_{ij} \tag{4}$$

Consider now the *static continuation* of the stress and body force fields beyond the domain V, Fig. 2(b), namely

$$\begin{aligned} \sigma^*_{ij}(\mathbf{x}^*) &= \sigma_{ij}(\mathbf{x}) + \sigma_{ij,k}\delta\varphi_k(\mathbf{x}), \\ f^*_i(\mathbf{x}^*) &= f_i(\mathbf{x}) + f_{i,k}\delta\varphi_k(\mathbf{x}). \end{aligned} \tag{5}$$

Such stress field satisfies the equilibrium conditions beyond the boundary S, thus

$$\sigma^*_{ij,j} + f^*_i = \sigma_{ij,j} + \sigma_{ij,kj}\delta\varphi_k + f_i + f_{i,k}\delta\varphi_k = 0. \tag{6}$$

The loaded boundary S_T is transformed into S^*_T and

$$T^*_i(\mathbf{x}^*) = \sigma^*_{ij}(\mathbf{x}^*)n^*_j, \tag{7}$$

where $\mathbf{n}^*$ is the unit normal vector to S^*_T. For the transformed configuration V^*, there is

$$\int \sigma^*_{ij}\epsilon^*_{ij}\,\mathrm{d}V^* = \int T^*_i u^0_i\,\mathrm{d}S_u + \int T^*_i u^*_i\,\mathrm{d}S^*_T + \int f^*_i u^*_i\,\mathrm{d}V^*. \tag{8}$$

Let us now transform the integration within V^* and over S^*_T to integration within V and over S_T. Since the following transformation rules occur, see Appendix A,

$$\mathrm{d}V^* = (1 + \delta\varphi_{k,k})\,\mathrm{d}V,$$

$$n^*_j\,\mathrm{d}S^* = (n_j + n_j\delta\varphi_{k,k} - n_k\delta\varphi_{k,j})\,\mathrm{d}S,$$

$$\delta n_j = n^*_j - n_j = n_j n_k n_l \delta\varphi_{k,l} - n_k\delta\varphi_{k,j}, \tag{9}$$

$$\delta(\mathrm{d}S) = (\delta\varphi_{k,k} - n_i\delta\varphi_{i,n})\,\mathrm{d}S, \tag{9}$$

where $\mathbf{n}$ denotes the unit normal vector to S_T, eqn (8) can be presented in the form

$$\begin{aligned} &\int (\sigma_{ij} + \sigma_{ij,k}\delta\varphi_k)(\epsilon_{ij} + \delta\bar{\epsilon}_{ij} + \epsilon_{ij,k}\delta\varphi_k)(1 + \delta\varphi_{k,k})\,\mathrm{d}V = \\ &= \int T_i u^0_i\,\mathrm{d}S_u + \int (\sigma_{ij} + \sigma_{ij,k}\delta\varphi_k)(u_i + \delta\bar{u}_i + u_{i,k}\delta\varphi_k) \\ &\times (n_j + n_j\delta\varphi_{k,k} - n_k\delta\varphi_{k,j})\,\mathrm{d}S_T + \int (f_i + f_{i,k}\delta\varphi_k)(u_i + \delta\bar{u}_i + u_{i,k}\delta\varphi_k)(1 + \delta\varphi_{k,k})\,\mathrm{d}V. \end{aligned} \tag{10}$$

Since for the untransformed configuration, there is

$$\int \sigma_{ij}\epsilon_{ij}\,\mathrm{d}V = \int T_i u_i^0\,\mathrm{d}S_u + \int T_i^0 u_i\,\mathrm{d}S_T + \int f_i u_i\,\mathrm{d}V, \tag{11}$$

after subtracting (11) from (10), one obtains the virtual displacement equation

$$\int \sigma_{ij}\delta\bar{\epsilon}_{ij}\,\mathrm{d}V = \int T_i^0\delta\bar{u}_i\,\mathrm{d}S_T + \int f_i\delta\bar{u}_i\,\mathrm{d}V + \int [(\sigma_{ik}\delta\varphi_j - \sigma_{ij}\delta\varphi_k)u_i]_{,j}n_k\,\mathrm{d}S_T. \tag{12}$$

Applying the Stoke's theorem to the last integral of (12), it can be reduced to the curvilinear integral along the curve bounding the surface portion S_T on S, thus

$$\int \sigma_{ij}\delta\bar{\epsilon}_{ij}\,\mathrm{d}V = \int T_i^0\delta\bar{u}_i\,\mathrm{d}S_T + \int f_i\delta\bar{u}_i\,\mathrm{d}V - \oint e_{jkl}\sigma_{ij}u_i t_k^{\Gamma}\delta\varphi_l^{\Gamma}\,\mathrm{d}\Gamma, \tag{13}$$

where t_k^{Γ} is the unit vector tangential to Γ, $\delta\varphi_l^{\Gamma}$ is the transformation variation on Γ and e_{jkl} is the permutation symbol. However, when $\delta\varphi_l^{\Gamma} = 0$ on Γ, the last term of (13) vanishes and there is

$$\int \boldsymbol{\sigma}\cdot\delta\bar{\boldsymbol{\epsilon}}\,\mathrm{d}V = \int \mathbf{T}^0\cdot\delta\bar{\mathbf{u}}\,\mathrm{d}S_T + \int \mathbf{f}\cdot\delta\bar{\mathbf{u}}\,\mathrm{d}V, \tag{14}$$

where the dot between two tensors or vectors denotes summation with respect to their indices.

This form of the virtual displacement equation will be used in subsequent analysis.

2.2 *Virtual stress equation*

Consider now the simultaneous variation of the stress, body force and transformation fields. Simultaneously as previously, we can write

$$\sigma_{ij}^*(\mathbf{x}^*) = \sigma_{ij}(\mathbf{x}) + \delta\sigma_{ij}(\mathbf{x}) = \sigma_{ij}(\mathbf{x}) + \delta\bar{\sigma}_{ij}(\mathbf{x}) + \sigma_{ij,k}(\mathbf{x})\delta\varphi_k,$$

$$f_i^*(\mathbf{x}^*) = f_i(\mathbf{x}) + \delta f_i(\mathbf{x}) = f_i(\mathbf{x}) + \delta\bar{f}_i(\mathbf{x}) + f_{i,k}(\mathbf{x})\delta\varphi_k, \tag{15}$$

where $\delta\bar{\sigma}_{ij} = \sigma_{ij}^*(\mathbf{x}) - \sigma_{ij}(\mathbf{x})$ and $\delta\bar{f}_i = f_i^*(\mathbf{x}) - f_i(\mathbf{x})$ are the stress and body force variations for the fixed configuration of the body. Since the stress σ_{ij}^* is statically admissible, thus

$$\sigma_{ij,j}^* + f_i^* = \sigma_{ij,j} + \delta\bar{\sigma}_{ij,j} + \sigma_{ij,jk}\delta\varphi_k + f_i + \delta\bar{f}_i + f_{i,k}\delta\varphi_k = 0, \tag{16}$$

and

$$\delta\bar{\sigma}_{ij,j} + \delta\bar{f}_i = 0 \quad \text{within } V. \tag{17}$$

Continuing analytically the displacement and strain fields from V to V^*, we can write

$$u_i^*(\mathbf{x}^*) = u_i(\mathbf{x}) + u_{i,k}(\mathbf{x})\delta\varphi_k,$$

$$\epsilon_{ij}^*(\mathbf{x}^*) = \epsilon_{ij}(\mathbf{x}) + \epsilon_{ij,k}(\mathbf{x})\delta\varphi_k. \tag{18}$$

Substituting (15)–(18) into (8), retransforming to the initial domain V and subtracting (11), the virtual stress equation takes the form

$$\int \delta\bar{\sigma}_{ij}\epsilon_{ij}\,\mathrm{d}V = \int \delta\bar{f}_i u_i\,\mathrm{d}V + \int \delta T_i u_i^0\,\mathrm{d}S_u + \int \delta\bar{\sigma}_{ij}u_i n_j\,\mathrm{d}S_T - \oint e_{jkl}\sigma_{ij}u_i t_k^{\Gamma}\delta\varphi_l^{\Gamma}\,\mathrm{d}\Gamma. \tag{19}$$

Let us note that for $\delta\varphi_l^{\Gamma} = 0$ on Γ, the last term of (19) vanishes.

Consider now the variation of surface tractions on the boundary portion S_T. Denoting the total variation of these tractions by $\delta\mathbf{T}^0$, we can write

$$\delta T_i^0 = T_i^*(\mathbf{x}^*) - T_i^0(\mathbf{x}) = \delta\sigma_{ij}\delta n_j + \sigma_{ij}\delta n_j, \tag{20}$$

and in view of (15) and the third equality of (9), we obtain from (20)

$$\delta\bar{\sigma}_{ij}n_j = \delta T_i^0 - T_i^0 n_k n_l \delta\varphi_{k,l} - \sigma_{ij,k}n_j\delta\varphi_k + \sigma_{ij}n_k\delta\varphi_{k,j}. \tag{21}$$

Using now (21) in (19), the virtual stress equations can be presented as follows

$$\int \delta\bar{\sigma}_{ij}\epsilon_{ij}\,\mathrm{d}V = \int \delta\bar{f}_i u_i\,\mathrm{d}V + \int \delta T_i u_i^0\,\mathrm{d}S_u + \int [\sigma_{ij}u_i(\delta_{jl} - n_l n_j)n_k\delta\varphi_{k,l} - \sigma_{ij,k}u_i n_j\delta\varphi_k]\,\mathrm{d}S_T + \int \delta T_i^0 u_i\,\mathrm{d}S_T, \tag{22}$$

where δ_{jl} denotes the Kronecker's symbol.

Let now the surface S_T be parametrized by an orthogonal curvilinear system α, β coinciding with the principal curvature lines on S_T and let a_k, b_k denote the unit vectors tangential to α and β. The transformation components in the coordinate system α, β, n are now expressed as follows

$$\delta\varphi_a = a_k\delta\varphi_k, \quad \delta\varphi_b = b_k\delta\varphi_k, \quad \delta\varphi_n = n_k\delta\varphi_k, \tag{23}$$

and since for any continuous and differentiable function $f(\mathbf{x})$ on S_T there is

$$f_{,k} = \frac{1}{A}f_{,\alpha}a_k + \frac{1}{B}f_{,\beta}b_k + f_{,n}n_k, \tag{24}$$

where A^2 and B^2 are the coefficients of the first quadratic form on S_T, eqn (22) can be presented in the form

$$\int \delta\bar{\sigma}_{ij}\epsilon_{ij}\,\mathrm{d}V = \int \delta T_i u_i^0\,\mathrm{d}S_u + \int \delta T_i^0 u_i\,\mathrm{d}S_T + \int \delta\bar{f}_i u_i\,\mathrm{d}V + \int f_i u_i n_k\delta\varphi_k\,\mathrm{d}S_T + \int \{[(T_i^0 u_i)_{,n} - 2T_i^0 u_i H - \sigma_{ij}\epsilon_{ij}]n_k - T^0_{i,k}u_i\}\delta\varphi_k\,\mathrm{d}S_T + \int [(T_i^0 u_i B a_k\delta\varphi_k)_{,\alpha} + (T_i^0 u_i A b_k\delta\varphi_k)_{,\beta}]\frac{1}{AB}\,\mathrm{d}S_T, \tag{25}$$

where H denotes the mean surface curvature on S_T, satisfying the equality

$$2Hn_k = \frac{1}{AB}[(Ba_k)_{,\alpha} + (Ab_k)_{,\beta}]. \tag{26}$$

Since the variations $\delta\varphi_k$ vanish on the curve Γ bounding the surface portion S_T undergoing transformation, then the last integral of (25) vanishes and the virtual stress equation is alternatively expressed as follows

$$\int \delta\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{\epsilon}\,\mathrm{d}V = \int \delta\,\mathbf{T}\cdot\mathbf{u}^0\,\mathrm{d}S_u + \int \delta\mathbf{T}^0\cdot\mathbf{u}\,\mathrm{d}S_T + \int \delta\bar{\mathbf{f}}\cdot\mathbf{u}\,\mathrm{d}V + \int \{[(\mathbf{T}^0\cdot\mathbf{u})_{,n} + \mathbf{f}\cdot\mathbf{u} - 2\mathbf{T}^0\cdot\mathbf{u}H - \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}]n_k - \mathbf{T}^0_{,k}\cdot\mathbf{u}\}\delta\varphi_k\,\mathrm{d}S_T. \tag{27}$$

Let us note that the eqns (22) and (27) are valid for both conservative and non-conservative loading.

In particular, when $\delta\varphi_k = 0$ on S_T and the boundary variation occurs only on the free boundary portion S_0 where $\mathbf{T}^0 = 0$, eqn (27) becomes

$$\int \delta\bar{\sigma}\cdot\epsilon\,\mathrm{d}V = \int \delta\mathbf{T}\cdot\mathbf{u}^0\,\mathrm{d}S_u + \int \delta\bar{\mathbf{f}}\cdot\mathbf{u}\,\mathrm{d}V - \int (\sigma\cdot\epsilon - \mathbf{f}\cdot\mathbf{u})\delta\varphi_n\,\mathrm{d}S_0. \tag{28}$$

Introducing a local Cartesian system y_k on S_0 with the axis y_3 normal to S_0 and the axes y_1, y_2 lying in the plane tangential to S_0, eqn (28) can be presented in an alternative form

$$\int \delta\bar{\sigma}\cdot\epsilon\,\mathrm{d}V = \int \delta\mathbf{T}\cdot\mathbf{u}^0\,\mathrm{d}S_u + \int \delta\bar{\mathbf{f}}\cdot\mathbf{u}\,\mathrm{d}V - \int (\sigma_{kl}\epsilon_{kl} - \mathbf{f}\cdot\mathbf{u})\delta\varphi_n\,\mathrm{d}S_0, \tag{29}$$

where σ_{kl} and ϵ_{kl} $(k, l = 1, 2)$ are "internal" stress and strain components referred to the axes lying within the tangent plane to S_0.

2.3 *Transformation of the supported boundary* S_u

Consider now the case when the transformation field modifies the boundary S_u on which the displacement vector $\mathbf{u} = \mathbf{u}^0$ is specified. Since now, see eqn (3), $\delta\mathbf{u} = \delta\bar{\mathbf{u}} + \mathbf{u}_{,k}\delta\varphi_k = 0$ on S_u, the virtual displacement equation takes the form

$$\int \sigma\cdot\delta\bar{\epsilon}\,\mathrm{d}V = \int \mathbf{f}\cdot\delta\bar{\mathbf{u}}\,\mathrm{d}V + \int \mathbf{T}^0\cdot\delta\mathbf{u}\,\mathrm{d}S_T - \int \mathbf{T}\cdot\mathbf{u}_{,k}\delta\varphi_k\,\mathrm{d}S_u, \tag{30}$$

provided the boundary variation vanishes on the curve Γ separating S_u from other boundary portions. On the other hand, the virtual stress equation is identical to (22) or (27), that is

$$\int \delta\bar{\sigma}\cdot\epsilon\,\mathrm{d}V = \int \delta\bar{\mathbf{f}}\cdot\mathbf{u}\,\mathrm{d}V + \int \delta\mathbf{T}\cdot\mathbf{u}^0\,\mathrm{d}S_u + \int \{[(\mathbf{T}\cdot\mathbf{u})_{,n} - 2\mathbf{T}\cdot\mathbf{u}^0 H$$
$$+ \mathbf{f}\cdot\mathbf{u} - \sigma\cdot\epsilon]n_k - \mathbf{T}_{,k}\cdot\mathbf{u}^0\}\delta\varphi_k\,\mathrm{d}S_u. \tag{31}$$

For the case of rigidly supported boundary, that is $\mathbf{u}^0 = 0$ on S_u, eqn (30) is reduced to the form

$$\int \sigma\cdot\delta\bar{\epsilon}\,\mathrm{d}V = \int \mathbf{f}\cdot\delta\bar{\mathbf{u}}\,\mathrm{d}V + \int \mathbf{T}^0\cdot\delta\bar{\mathbf{u}}\,\mathrm{d}S_T - \int \mathbf{T}\cdot\mathbf{u}_{,n}\delta\varphi_n\,\mathrm{d}S_u, \tag{32}$$

whereas the virtual stress equation (31) takes the form

$$\int \delta\bar{\sigma}\cdot\epsilon\,\mathrm{d}V = \int \delta\bar{\mathbf{f}}\cdot\mathbf{u}\,\mathrm{d}V. \tag{33}$$

We have thus discussed consecutively the variation of each external boundary portion, assuming that there is *no interaction* between variations of these portions. As mentioned previously, the case of coupled shape variations will be treated separately.

2.4 *Transformations of the interface* S_c

Consider now a two-phase body composed of two materials occupying subdomains V_1 and V_2 and separated by the interface S_c, that is $V = V_1 \cup V_2$, Fig. 1(b). Assume the transformation field to modify only S_c, whereas the external boundary remains unchanged, thus $\delta\boldsymbol{\varphi} = \mathbf{0}$ on S.

Since the stiffness moduli vary discontinuously on S_c, the displacements $\mathbf{u} = \mathbf{u}^c$ and the surface tractions $\mathbf{T}^c = \sigma^c\cdot\mathbf{n}^c$ are continuous, but their gradients and stress components exhibit discontinuities. Denoting by $[\![\;\;]\!]$ the discontinuity of the enclosed quantity on S_c

calculated as a difference of respective values in the domains V_1 and V_2, we have

$$[\![\mathbf{u}^c]\!] = 0, \qquad [\![\mathbf{T}^c]\!] = [\![\boldsymbol{\sigma}]\!]\cdot\mathbf{n}^c,$$
$$[\![\mathbf{u}^c_{,k}]\!] = [\![\mathbf{u}^c_{,n}]\!]n_k{}^c, \quad [\![\mathbf{T}^c_{,k}]\!] = [\![\mathbf{T}^c_{,n}]\!]n_k{}^c. \tag{34}$$

Let $\mathbf{f}_1, \boldsymbol{\sigma}_1, \boldsymbol{\epsilon}_1, \mathbf{u}_1$ and $\mathbf{f}_2, \boldsymbol{\sigma}_2, \boldsymbol{\epsilon}_2, \mathbf{u}_2$ be body forces, stress, strain and displacement fields within V_1 and V_2. The virtual displacement equation for the field $\delta\bar{\mathbf{u}}$ now takes the form

$$\int \boldsymbol{\sigma}_1\cdot\delta\bar{\boldsymbol{\epsilon}}_1\,\mathrm{d}V_1 + \int \boldsymbol{\sigma}_2\cdot\delta\bar{\boldsymbol{\epsilon}}_2\,\mathrm{d}V_2 = \int \mathbf{f}_1\cdot\delta\bar{\mathbf{u}}_1\,\mathrm{d}V_1 + \int \mathbf{f}_2\cdot\delta\bar{\mathbf{u}}_2\,\mathrm{d}V_2$$
$$+ \int \mathbf{T}^0\cdot\delta\bar{\mathbf{u}}\,\mathrm{d}S_T + \int \mathbf{T}^c\cdot[\![\delta\bar{\mathbf{u}}^c]\!]\,\mathrm{d}S_c. \tag{35}$$

Since $[\![\delta\mathbf{u}^c]\!]$ is continuous on S_c, we have

$$[\![\delta\mathbf{u}^c]\!] = [\![\delta\bar{\mathbf{u}}^c]\!] + [\![\mathbf{u}^c_{,n}]\!]\delta\varphi_n = 0. \tag{36}$$

Substituting (36) into (35), we obtain

$$\int \boldsymbol{\sigma}_1\cdot\delta\bar{\boldsymbol{\epsilon}}_1\,\mathrm{d}V_1 + \int \boldsymbol{\sigma}_2\cdot\delta\bar{\boldsymbol{\epsilon}}_2\,\mathrm{d}V_2 = \int \mathbf{f}_1\cdot\delta\bar{\mathbf{u}}_1\,\mathrm{d}V_1 + \int \mathbf{f}_2\cdot\delta\bar{\mathbf{u}}_2\,\mathrm{d}V_2$$
$$+ \int \mathbf{T}^0\cdot\delta\bar{\mathbf{u}}\,\mathrm{d}S_T - \int \mathbf{T}^c\cdot[\![\mathbf{u}^c_{,n}]\!]\delta\varphi_n\,\mathrm{d}S_c, \tag{37}$$

provided $\delta\boldsymbol{\varphi}^\Gamma = 0$ on the curve Γ lying on the boundary surface S. The derivation of virtual stress equation follows similar lines. In fact, we may write

$$\int \delta\bar{\boldsymbol{\sigma}}_1\cdot\boldsymbol{\epsilon}_1\,\mathrm{d}V_1 + \int \delta\bar{\boldsymbol{\sigma}}_2\cdot\boldsymbol{\epsilon}_2\,\mathrm{d}V_2 = \int \delta\bar{\mathbf{f}}_1\cdot\mathbf{u}_1\,\mathrm{d}V_1 + \int \delta\bar{\mathbf{f}}_2\cdot\mathbf{u}_2\,\mathrm{d}V_2$$
$$+ \int \delta\mathbf{T}\cdot\mathbf{u}^0\,\mathrm{d}S_u + \int \mathbf{u}^c\cdot[\![\delta\bar{\boldsymbol{\sigma}}]\!]\cdot\mathbf{n}^c\,\mathrm{d}S_c, \tag{38}$$

and since $\delta\mathbf{T}$ is continuous on S_c, thus

$$[\![\delta\mathbf{T}]\!] = [\![\boldsymbol{\sigma}]\!]\cdot\delta\mathbf{n}^c + [\![\delta\boldsymbol{\sigma}]\!]\cdot\mathbf{n}^c = [\![\boldsymbol{\sigma}]\!]\cdot\delta\mathbf{n}^c + [\![\delta\bar{\boldsymbol{\sigma}}]\!]\cdot\mathbf{n}^c + [\![\boldsymbol{\sigma}_{,k}]\!]\delta\varphi_k\cdot\mathbf{n}^c = 0, \tag{39}$$

and in view of (9), there is

$$[\![\delta\bar{\sigma}_{ij}]\!]n_j^c = [\![\sigma_{ij}]\!]n_k{}^c\delta\varphi_{k,j} - [\![\sigma_{ij,k}]\!]n_j{}^c\delta\varphi_k. \tag{40}$$

Substituting (40) into (38), it follows that virtual stress equation takes the form

$$\int \delta\bar{\boldsymbol{\sigma}}_1\cdot\boldsymbol{\epsilon}_1\,\mathrm{d}V_1 + \int \delta\bar{\boldsymbol{\sigma}}_2\cdot\boldsymbol{\epsilon}_2\,\mathrm{d}V_2 = \int \delta\bar{\mathbf{f}}_1\cdot\mathbf{u}_1\,\mathrm{d}V_1 + \int \delta\bar{\mathbf{f}}_2\cdot\mathbf{u}_2\,\mathrm{d}V_2$$
$$+ \int \delta\mathbf{T}\cdot\mathbf{u}^0\,\mathrm{d}S_u + \int ([\![\sigma_{ij}]\!]u_i{}^c n_k{}^c\delta\varphi_{k,j} - [\![\sigma_{ij,k}]\!]u_i{}^c n_j{}^c\delta\varphi_k)\,\mathrm{d}S_c, \tag{41}$$

or alternatively by applying the Stoke's theorem, it is obtained for $\delta\boldsymbol{\varphi}^\Gamma = 0$ on Γ

$$\int \delta\bar{\boldsymbol{\sigma}}_1\cdot\boldsymbol{\epsilon}_1\,\mathrm{d}V_1 + \int \delta\bar{\boldsymbol{\sigma}}_2\cdot\boldsymbol{\epsilon}_2\,\mathrm{d}V_2 = \int \delta\bar{\mathbf{f}}_1\cdot\mathbf{u}_1\,\mathrm{d}V_1 + \int \delta\bar{\mathbf{f}}_2\cdot\mathbf{u}_2\,\mathrm{d}V_2$$
$$+ \int \delta\mathbf{T}\cdot\mathbf{u}^0\,\mathrm{d}S_u + \int ([\![\mathbf{f}]\!]\cdot\mathbf{u} - [\![\sigma_{kl}]\!]\epsilon^c_{kl})\delta\varphi_n\,\mathrm{d}S_c, \tag{42}$$

where σ_{kl} and ϵ_{kl} denote the 'internal' stress and strain components referred to the coordinate axes lying in the plane tangential to S_c. A more detailed discussion of the interface variation is presented in[4].

We have thus discussed consecutively the virtual equations for the variation of each boundary portion. These equations will be now used in the subsequent analysis.

3. VARIATION OF POTENTIAL AND COMPLEMENTARY ENERGIES ASSOCIATED WITH BOUNDARY VARIATION

3.1 *General case*

In this Section, we shall apply the virtual displacement and stress equations in order to derive the first variations of the potential and complementary energies corresponding to transformation of the external boundary or the interface of a structure. Such variations can next be used in deriving the optimality conditions for an optimal design with specified global elastic compliance.

Consider the potential energy

$$\Pi_u(\mathbf{u}, \mathbf{f}, \mathbf{T}^0, \varphi) = \int U(\epsilon)\,\mathrm{d}V - \int \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}V - \int \mathbf{T}^0\cdot\mathbf{u}\,\mathrm{d}S_T, \tag{43}$$

where $U(\epsilon)$ denotes the specific strain energy per unit volume. Our analysis in this Section will be referred to both *linear* and *non-linear* elastic materials. The first variation of Π_u equals

$$\delta\Pi_u = \int \frac{\partial U}{\partial \epsilon}\cdot\delta\bar{\epsilon}\,\mathrm{d}V - \int \mathbf{f}\cdot\delta\bar{\mathbf{u}}\,\mathrm{d}V + \int (U(\epsilon) - \mathbf{f}\cdot\mathbf{u})\delta\varphi_n\,\mathrm{d}S_T - \delta\int \mathbf{T}^0\cdot\mathbf{u}\,\mathrm{d}S_T, \tag{44}$$

and since

$$\delta\int \mathbf{T}^0\cdot\mathbf{u}\,\mathrm{d}S_T = \int \delta\mathbf{T}^0\cdot\mathbf{u}\,\mathrm{d}S_T + \int \mathbf{T}^0\cdot\delta\mathbf{u}\,\mathrm{d}S_T + \int \mathbf{T}^0\cdot\mathbf{u}\delta(\mathrm{d}S_T), \tag{45}$$

in view of (9) and using the virtual displacement equation (14), we obtain

$$\delta\Pi_u = \int \{[(U - \mathbf{f}\cdot\mathbf{u})n_k - \mathbf{T}^0\cdot\mathbf{u}_{,k}]\delta\varphi_k - \mathbf{T}^0\cdot\mathbf{u}(\delta_{kl} - n_k n_l)\delta\varphi_{k,l}\}\,\mathrm{d}S_T - \int \delta\mathbf{T}^0\cdot\mathbf{u}\,\mathrm{d}S_T, \tag{46}$$

or in an alternative form

$$\delta\Pi_u = \int \{[U - \mathbf{f}\cdot\mathbf{u} - (\mathbf{T}^0\cdot\mathbf{u})_{,n} + 2\mathbf{T}^0\cdot\mathbf{u}H]n_k + \mathbf{T}^0_{,k}\cdot\mathbf{u}\}\delta\varphi_k\,\mathrm{d}S_T - \int \delta\mathbf{T}^0\cdot\mathbf{u}\,\mathrm{d}S_T, \tag{47}$$

where it was assumed that $\delta\varphi^\Gamma = 0$ on the curve Γ bounding the varying portion of the boundary S_T.

Assume first that the surface loading is conservative and does not depend on the surface configuration, but may vary with a position. Then $\delta\mathbf{T}^0 = \mathbf{T}^0_{,k}\delta\varphi_k$ and (47) becomes

$$\delta\Pi_u = \int [U - \mathbf{f}\cdot\mathbf{u} - (\mathbf{T}^0\cdot\mathbf{u})_{,n} + 2\mathbf{T}^0\cdot\mathbf{u}H]\delta\varphi_n\,\mathrm{d}S_T. \tag{48}$$

As an example of a non-conservative loading, consider the pressure loading

$$\mathbf{T}^0 = p(\mathbf{x})\mathbf{n} \tag{49}$$

directed along the normal to the surface. In view of (9), there is

$$\delta T_i^0 = \delta p(\mathbf{x})n_i + p(\mathbf{x})\delta n_i = p_{,k}n_i\delta\varphi_k + p(n_i n_k\delta\varphi_{k,n} - n_k\delta\varphi_{k,i}), \tag{50}$$

and the variation $\delta\Pi_u$ is expressed in the form

$$\delta\Pi_u = \int [U - \mathbf{f}\cdot\mathbf{u} - \operatorname{div}(p\mathbf{u})]\delta\varphi_n \, dS_T. \tag{51}$$

Consider now the complementary energy

$$\Pi_\sigma(\sigma, \mathbf{T}, \varphi) = \int W(\sigma)\, dV - \int \mathbf{T}\cdot\mathbf{u}^0 \, dS_u, \tag{52}$$

where $W(\sigma)$ denotes the specific stress energy per unit material volume. The first variation of Π_σ equals

$$\delta\Pi_\sigma = \int \frac{\partial W}{\partial \sigma}\cdot\delta\bar{\sigma}\, dV + \int W n_k \delta\varphi_k \, dS_T - \int \delta\mathbf{T}\cdot\mathbf{u}^0 \, dS_u, \tag{53}$$

and in view of the virtual stress equation (22), in which it is now assumed that there is no local variation of body forces, it follows that

$$\delta\Pi_\sigma = \int [(W n_k - \sigma_{ij,k} u_i n_j)\delta\varphi_k + \sigma_{ij} u_i(\delta_{jl} - n_j n_l) n_k \delta\varphi_{k,l}]\, dS_T + \int \delta\mathbf{T}^0\cdot\mathbf{u}\, dS_T, \tag{54}$$

or alternatively

$$\delta\Pi_\sigma = \int \{[W + (\mathbf{T}^0\cdot\mathbf{u})_{,n} - 2\mathbf{T}^0\cdot\mathbf{u}H + \mathbf{f}\cdot\mathbf{u} - \sigma\cdot\epsilon]n_k - \mathbf{T}^0_{,k}\cdot\mathbf{u}\}\delta\varphi_k \, dS_T + \int \delta\mathbf{T}^0\cdot\mathbf{u}\, dS_T. \tag{55}$$

In particular, for a loading independent of surface configuration, (55) provides

$$\delta\Pi_\sigma = \int [W + (\mathbf{T}^0\cdot\mathbf{u})_{,n} - 2\mathbf{T}^0\cdot\mathbf{u}H + \mathbf{f}\cdot\mathbf{u} - \sigma\cdot\epsilon]\delta\varphi_n \, dS_T, \tag{56}$$

whereas in the case of pressure loading (49), there is

$$\delta\Pi_\sigma = \int [W + \mathbf{f}\cdot\mathbf{u} - \sigma\cdot\epsilon + \operatorname{div}(p\mathbf{u})]\delta\varphi_n \, dS_T. \tag{57}$$

When only free boundary S_0 is subject to variation, and there are no body forces, the derived expressions for first variations are considerably simplified. In fact from (47) and (55) it follows that after setting $\mathbf{T}^0 = 0$, $\mathbf{f} = 0$ one obtains

$$\delta\Pi_u = \int U\delta\varphi_n \, dS_0, \quad \delta\Pi_\sigma = \int (W - \sigma\cdot\epsilon)\delta\varphi_n \, dS_0 = -\int U\delta\varphi_n \, dS_0. \tag{58}$$

Thus, the potential energy of a structure increases and the complementary energy decreases by moving the free boundary in the exterior, that is by adding the material to a structure.

When, on the other hand, only supported boundary S_u is modified, the first variations of (43) and (52), in view of (30) and (31), take the form

$$\delta\Pi_u = \int [(U - \mathbf{f}\cdot\mathbf{u})n_k - \mathbf{T}\cdot\mathbf{u}_{,k}]\delta\varphi_k \, dS_u, \tag{59}$$

and

$$\delta\Pi_\sigma = \int [(W + \mathbf{f}\cdot\mathbf{u} - \sigma\cdot\epsilon)n_k + \mathbf{T}\cdot\mathbf{u}_{,k}]\delta\varphi_k \, dS_u. \tag{60}$$

For the variation of the interface S_c, in view of (37) and (42), we obtain

$$\delta \Pi_u = \int ([\![U]\!] - [\![\mathbf{f}]\!] \cdot \mathbf{u}^c - \mathbf{T}^c \cdot [\![\mathbf{u}^c_{,n}]\!]) \delta\varphi_n \, \mathrm{d}S_c, \tag{61}$$

and

$$\delta \Pi_\sigma = \int ([\![W]\!] + [\![\mathbf{f}]\!] \cdot \mathbf{u}^c - [\![\sigma_{kl}]\!] \epsilon^c_{kl}) \delta\varphi_n \, \mathrm{d}S_c, \tag{62}$$

where σ_{kl} and ϵ_{kl} denote the 'internal' stress and strain components within the plane tangential to S_c. The relation (61) was earlier derived in a different way by Eshelby[8] for the case of translation of the interface.

3.2 *Translation and rotation of the boundary*

Consider now the body bounded by a surface S_T on which the fixed surface tractions $\mathbf{T}^0$ are prescribed. Assume, for simplicity, that the body forces are neglected and consider two particular cases of shape transformation, namely (i) translation and (ii) rotation of the body with external forces $\mathbf{T}^0$ being respectively translated and rotated.

In the case of translation of the boundary by a vector $\delta\mathbf{a}$, it can be written

$$\begin{aligned} &\delta\varphi_k(\mathbf{x}) = \delta a_k = \text{const.} \\ &\delta \mathbf{T}^0(\mathbf{x}) = 0 \end{aligned} \qquad \text{for } \mathbf{x} \in S_T, \tag{63}$$

and from (46) and (54) we obtain the variation of Π_u and Π_σ, expressed as follows

$$\delta \Pi_u = \int (U n_k - \mathbf{T}^0 \cdot \mathbf{u}_{,k}) \, \mathrm{d}S_T \delta a_k = \int (U \delta_{jk} - \sigma_{ij} u_{i,k}) n_j \, \mathrm{d}S_T \delta a_k, \tag{64}$$

and

$$\delta \Pi_\sigma = \int (W n_k - \sigma_{ij,k} u_i n_j) \, \mathrm{d}S_T \delta a_k = \int (W \delta_{jk} - \sigma_{ij,k} u_i) n_j \, \mathrm{d}S_T \delta a_k. \tag{65}$$

On the other hand, when the body is rotated around a point R by the infinitesimal rotation vector $\delta\omega$, the variations of point positions on S_T and their spatial derivatives are

$$\begin{aligned} &\delta\varphi_j(\mathbf{x}) = e_{jkl} x_l \delta\omega_k, \\ &\delta\varphi_{j,p} = e_{jkl} \delta_{lp} \delta\omega_k = e_{jkp} \delta\omega_k. \end{aligned} \tag{66}$$

Similarly, the variation of surface traction due to rotation of the traction vector takes the form

$$\delta T^0_j = e_{jkl} T^0_l \delta\omega_k. \tag{67}$$

Substituting (66) and (67) into (46) and (54), we obtain

$$\begin{aligned} \delta \Pi_u = &\int e_{jkl} [(U n_j - \mathbf{T}^0 \cdot \mathbf{u}_{,j}) x_l - \mathbf{T}^0 \cdot \mathbf{u} (\delta_{jl} - n_j n_l)] \, \mathrm{d}S_T \delta\omega_k \\ &- \int e_{jkl} T^0_l u_j \, \mathrm{d}S_T \delta\omega_k = \int e_{jkl} (U x_l n_j + T^0_j u_l - T^0_i u_{i,j} x_l) \, \mathrm{d}S_T \delta\omega_k. \end{aligned} \tag{68}$$

and

$$\delta\Pi_\sigma = \int e_{jkl}[(Wn_j - \sigma_{ip,j}u_in_p)x_l + \sigma_{ip}u_i(\delta_{pl} - n_pn_l)n_j]\,\mathrm{d}S_T\delta\omega_k + \int e_{jkl}T_l^0u_j\,\mathrm{d}S_T\delta\omega_k$$

$$= \int e_{jkl}[Wx_ln_j - (\sigma_{jp}u_l + \sigma_{ip,j}u_ix_l)n_p + \delta_{il}u_in_j]\,\mathrm{d}S_T\delta\omega_k. \tag{69}$$

Let us note that the conservation laws formulated by Eshelby[8], Knowles and Sternberg[9] and Bui[11] can be derived from these expressions. Namely, considering the invariance of Π_u under translation and rotation of the domain of a homogeneous and isotropic body, we can write

$$\delta\Pi_u = 0. \tag{70}$$

In view of (64), the Eshelby conservation law for a homogeneous body takes the form

$$J_k = \int (U\delta_{jk} - \sigma_{ij}u_{i,k})n_j\,\mathrm{d}S = 0. \tag{71}$$

In the case of rotation (66), from (68) we obtain for an isotropic body

$$L_k = \int e_{jkl}(Ux_ln_j + T_j^0u_l - T_i^0u_{i,j}x_l)\,\mathrm{d}S = 0, \tag{72}$$

which is equivalent with that derived by Knowles and Sternberg[9]. Similarly, considering the stress energy, we arrive at the conservation law for a homogeneous body considered by Bui[11]

$$B_k = \int (W\delta_{jk} - \sigma_{ij,k}u_i)n_j\,\mathrm{d}S = 0. \tag{73}$$

Let us note that S can now be identified with any closed surface within the body, in particular with the boundary surface S_T.

4. VARIATION OF ARBITRARY STRESS, STRAIN OR DISPLACEMENT AND TRACTION FUNCTIONALS ASSOCIATED WITH BOUNDARY VARIATION

In this Section, we shall derive the expressions for first variations of the functionals G_1 and G_2 defined in Part I[1], corresponding to variations of the boundaries S_T, S_0, S_u and S_c. Similarly as in Section 2, let us denote the total variations of stress, strain and displacement by $\delta\sigma$, $\delta\epsilon$ and $\delta\mathbf{u}$, whereas the variations for fixed configuration are respectively $\delta\bar{\sigma}$, $\delta\bar{\epsilon}$ and $\delta\bar{\mathbf{u}}$.

4.1 *General case*

Similarly as in[1], consider the functional

$$G_1 = \int \Psi(\sigma)\,\mathrm{d}V + \int h(\mathbf{u})\,\mathrm{d}V + \int f(\mathbf{T})\,\mathrm{d}S_u + \int g(\mathbf{u})\,\mathrm{d}S_T, \tag{74}$$

and its first variation associated with the variation of the loaded boundary S_T

$$\delta G_1 = \int \frac{\partial \Psi}{\partial \sigma}\cdot\delta\bar{\sigma}\,\mathrm{d}V + \int \Psi n_k\delta\varphi_k\,\mathrm{d}S_T + \int \frac{\partial h}{\partial \mathbf{u}}\cdot\delta\bar{\mathbf{u}}\,\mathrm{d}V + \int hn_k\delta\varphi_k\,\mathrm{d}S_T$$

$$+ \int \frac{\partial \mathbf{f}}{\partial \mathbf{T}}\cdot\delta\mathbf{T}\,\mathrm{d}S_u + \int \left[\frac{\partial g}{\partial \mathbf{u}}\cdot\delta\bar{\mathbf{u}} + \frac{\partial g}{\partial \mathbf{u}}\cdot\mathbf{u}_{,k}\delta\varphi_k + g(\delta_{kl} - n_kn_l)\delta\varphi_{k,l}\right]\mathrm{d}S_T. \tag{75}$$

Following[1], consider now the adjoint structure of the same shape but satisfying the boundary conditions

$$\mathbf{T}^{a_0} = \frac{\partial g}{\partial \mathbf{u}} \text{ on } S_T, \quad \mathbf{u}^{a_0} = -\frac{\partial \mathrm{f}}{\partial \mathbf{T}} \text{ on } S_u, \quad \mathbf{f}^a = \frac{\partial \mathrm{h}}{\partial \mathbf{u}} \text{ within } V \tag{76}$$

and with the imposed initial strain field ϵ^i specified by

$$\epsilon^i = \frac{\partial \Psi}{\partial \sigma} \text{ within } V. \tag{77}$$

Denoting the stress within the adjoint structure by σ^r, its total strain field ϵ^a can be presented as a sum

$$\epsilon^a = \epsilon^i + \epsilon^r, \tag{78}$$

and it is compatible with the displacement field $\mathbf{u}^a$. The stress field σ^r is related to ϵ^r by Hooke's law, $\sigma^r = \mathbf{D} \cdot \epsilon^r$, and satisfies both equilibrium and boundary conditions

$$\operatorname{div} \sigma^r + \mathbf{f}^a = 0 \quad \text{within } V, \quad \sigma^r \cdot \mathbf{n} = \mathbf{T}^{a_0} \text{ on } S_T. \tag{79}$$

We therefore can write

$$\int \frac{\partial \Psi}{\partial \sigma} \cdot \delta\bar{\sigma} \, \mathrm{d}V = \int \epsilon^a \cdot \delta\bar{\sigma} \, \mathrm{d}V - \int \epsilon^r \cdot \delta\bar{\sigma} \, \mathrm{d}V, \tag{80}$$

and since

$$\int \epsilon^r \cdot \delta\bar{\sigma} \, \mathrm{d}V = \int \epsilon^r \cdot \mathbf{D} \cdot \delta\bar{\epsilon} \, \mathrm{d}V = \int \sigma^r \cdot \delta\bar{\epsilon} \, \mathrm{d}V = \int \mathbf{T}^{a_0} \cdot \delta\bar{\mathbf{u}} \, \mathrm{d}S_T + \int \mathbf{f}^a \cdot \delta\bar{\mathbf{u}} \, \mathrm{d}V, \tag{81}$$

in view of the virtual stress equation (22), in which it is assumed that there is no local variation of body forces, we obtain

$$\int \frac{\partial \Psi}{\partial \sigma} \cdot \delta\bar{\sigma} \, \mathrm{d}V = \int \delta\mathbf{T} \cdot \mathbf{u}^{a_0} \, \mathrm{d}S_u + \int \delta\mathbf{T}^0 \cdot \mathbf{u}^a \, \mathrm{d}S_T + \int [\sigma_{ij} u_i^a (\delta_{jl} - n_j n_l) n_k \delta\varphi_{k,l}$$

$$- \sigma_{ij,k} u_i^a n_j \delta\varphi_k] \, \mathrm{d}S_T - \int \mathbf{T}^{a_0} \cdot \delta\bar{\mathbf{u}} \, \mathrm{d}S_T - \int \mathbf{f}^a \cdot \delta\bar{\mathbf{u}} \, \mathrm{d}V, \tag{82}$$

and the first variation of G_1 is expressed as follows

$$\delta G_1 = \int \delta\mathbf{T}^0 \cdot \mathbf{u}^a \, \mathrm{d}S_T + \int \{[(\Psi + h) n_k + (\sigma_{ij}^r u_{i,k} - \sigma_{ij,k} u_i^a) n_j] \delta\varphi_k$$

$$+ [\sigma_{ij} u_i^a (\delta_{jl} - n_j n_l) n_k + g(\delta_{kl} - n_k n_l)] \delta\varphi_{k,l}\} \, \mathrm{d}S_T. \tag{83}$$

The alternative form of the first variation of G_1, in view of stress equation (27), can be expressed as follows

$$\delta G_1 = \int \delta\mathbf{T}^0 \cdot \mathbf{u}^a \, \mathrm{d}S_T + \int \{[\Psi + h + g_{,n} + (\mathbf{T}^0 \cdot \mathbf{u}^a)_{,n} - 2(g + \mathbf{T}^0 \cdot \mathbf{u}^a) H$$

$$- \sigma \cdot \epsilon^a + \mathbf{f} \cdot \mathbf{u}^a] n_k - \mathbf{T}^0_{,k} \cdot \mathbf{u}^a\} \delta\varphi_k \, \mathrm{d}S_T. \tag{84}$$

When only free boundary S_0 is subject to variation and there are no body forces, assuming

$g(\mathbf{u}) = 0$ and $h(\mathbf{u}) = 0$, the expression (84) becomes

$$\delta G_1 = \int (\Psi - \sigma_{kl}\epsilon^a_{kl})\delta\varphi_n \, dS_0, \tag{85}$$

where σ_{kl} and ϵ^a_{kl} denote the 'internal' stress and strain components of primary and adjoint structures within the plane tangential to S_0.

For the case of supported boundary variation, with S_T fixed, the first variation of G_1, in view of (30) and (31), takes the form

$$\delta G_1 = \int \{[\Psi + h + f_{,n} + (\mathbf{T}\cdot\mathbf{u}^{a0})_{,n} - 2(f + \mathbf{T}\cdot\mathbf{u}^{a0})H - \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^a$$
$$+ \mathbf{f}\cdot\mathbf{u}^{a0}]n_k + \mathbf{T}^a\cdot\mathbf{u}_{,k}\}\delta\varphi_k \, dS_u, \tag{86}$$

where static and kinematic fields accompanied with the adjoint structure satisfy once again the conditions (76)–(79). Consider now the functional

$$G_2 = \int \Phi(\epsilon)\, dV + \int h(\mathbf{u})\, dV + \int f(\mathbf{T})\, dS_u + \int g(\mathbf{u})\, dS_T, \tag{87}$$

and its first variation, expressed as follows

$$\delta G_2 = \int \frac{\partial\Phi}{\partial\epsilon}\cdot\delta\bar{\epsilon}\, dV + \int \Phi n_k\delta\varphi_k\, dS_T + \int \frac{\partial h}{\partial\mathbf{u}}\cdot\delta\bar{\mathbf{u}}\, dV + \int hn_k\delta\varphi_k\, dS_T$$
$$+ \int \frac{\partial f}{\partial\mathbf{T}}\cdot\delta\mathbf{T}\, dS_u + \int\left[\frac{\partial g}{\partial\mathbf{u}}\cdot\delta\bar{\mathbf{u}} + \frac{\partial g}{\partial\mathbf{u}}\cdot\mathbf{u}_{,k}\delta\varphi_k + g(\delta_{kl} - n_k n_l)\delta\varphi_{k,l}\right] dS_T. \tag{88}$$

Following[1], let us introduce the adjoint structure satisfying the boundary conditions (76) and with the imposed initial stress field

$$\boldsymbol{\sigma}^i = \frac{\partial\Phi}{\partial\epsilon} \quad \text{within } V. \tag{89}$$

Thus, we obtain

$$\int \frac{\partial\Phi}{\partial\epsilon}\cdot\delta\bar{\epsilon}\, dV = \int \boldsymbol{\sigma}^a\cdot\delta\bar{\epsilon}\, dV - \int \mathbf{T}^{a0}\cdot\delta\bar{\mathbf{u}}\, dS_T - \int \mathbf{f}^a\cdot\delta\bar{\mathbf{u}}\, dV, \tag{90}$$

and in view of stress equation (22), in which it is again assumed that there is no local variation of body forces, there is

$$\int \boldsymbol{\sigma}^a\cdot\delta\bar{\epsilon}\, dV = \int \boldsymbol{\sigma}^a\cdot\mathbf{E}\cdot\delta\bar{\boldsymbol{\sigma}}\, dV = \int \boldsymbol{\epsilon}^a\cdot\delta\bar{\boldsymbol{\sigma}}\, dV = \int \delta\mathbf{T}\cdot\mathbf{u}^{a0}\, dS_u$$
$$+ \int \delta\mathbf{T}^0\cdot\mathbf{u}^a\, dS_T + \int [\sigma_{ij}u^a_i(\delta_{jl} - n_j n_l)n_k\delta\varphi_{k,l} - \sigma_{ij,k}u^a_i n_j\delta\varphi_k]\, dS_T, \tag{91}$$

where $\boldsymbol{\sigma}$ and $\mathbf{T}^0$ are the stress and traction fields of the primary structure whereas $\mathbf{u}^a$ and $\boldsymbol{\epsilon}^a$ denote the displacement and strain fields of the adjoint structure. The variation δG_2 is now expressed in the form

$$\delta G_2 = \int \delta\mathbf{T}^0\cdot\mathbf{u}^a\, dS_T + \int \{[(\Phi + h)n_k + (\sigma^i_{ij}u_{i,k} - \sigma_{ij,k}u^a_i)n_j]\delta\varphi_k$$
$$+ [\sigma_{ij}u^a_i(\delta_{jl} - n_j n_l)n_k + g(\delta_{kl} - n_k n_l)]\delta\varphi_{k,l}\}\, dS_T, \tag{92}$$

or in an alternative form

$$\delta G_2 = \int \delta \mathbf{T}^0 \cdot \mathbf{u}^a \, dS_T + \int \{[\Phi + h + g_{,n} + (\mathbf{T}^0 \cdot \mathbf{u}^a)_{,n} - 2(g + \mathbf{T}^0 \cdot \mathbf{u}^a)H$$
$$- \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^a + \mathbf{f} \cdot \mathbf{u}^a] n_k - \mathbf{T}^0_{,k} \cdot \mathbf{u}^a\} \delta\varphi_k \, dS_T. \tag{93}$$

In particular, when $g(\mathbf{u}) = 0$, $h(\mathbf{u}) = 0$, there are no body forces and only free boundary S_0 is subject to variation, the expression (93) becomes

$$\delta G_2 = \int (\Phi - \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^a) \delta\varphi_n \, dS_0. \tag{94}$$

For the case of supported boundary variation, with S_T fixed, the first variation of G_2, in view of (30) and (31), equals

$$\delta G_2 = \int \{[\phi + h + f_{,n} + (\mathbf{T} \cdot \mathbf{u}^{a0})_{,n} - 2(f + \mathbf{T} \cdot \mathbf{u}^{a0})H - \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^a + \mathbf{f} \cdot \mathbf{u}^{a0}] n_k$$
$$+ \mathbf{T}^a \cdot \mathbf{u}_{,k}\} \delta\varphi_k \, dS_u. \tag{95}$$

Consider finally the case when only the interface S_c undergoes the variation. Consider the functional

$$G_1 = \int \Psi_1(\boldsymbol{\sigma}_1) \, dV_1 + \int \Psi_2(\boldsymbol{\sigma}_2) \, dV_2 + \int h_1(\mathbf{u}) \, dV_1 + \int h_2(\mathbf{u}) \, dV_2$$
$$+ \int f(\mathbf{T}) \, dS_u + \int g(\mathbf{u}) \, dS_T, \tag{96}$$

where Ψ_1 and Ψ_2, h_1 and h_2 are continuous and differentiable functions of stress and displacement within the domains V_1 and V_2 separated by the interface S_c. The variation of G_1 equals

$$\delta G_1 = \int \frac{\partial \Psi_1}{\partial \boldsymbol{\sigma}_1} \cdot \delta\bar{\boldsymbol{\sigma}}_1 \, dV_1 + \int \frac{\partial \Psi_2}{\partial \boldsymbol{\sigma}_2} \cdot \delta\bar{\boldsymbol{\sigma}}_2 \, dV_2 + \int [\![\Psi]\!] \delta\varphi_n \, dS_c + \int \frac{\partial h_1}{\partial \mathbf{u}} \cdot \delta\bar{\mathbf{u}} \, dV_1$$
$$+ \int \frac{\partial h_2}{\partial \mathbf{u}} \cdot \delta\bar{\mathbf{u}} \, dV_2 + \int [\![h]\!] \delta\varphi_n \, dS_c + \int \frac{\partial \mathbf{f}}{\partial \mathbf{T}} \cdot \delta\mathbf{T} \, dS_u + \int \frac{\partial \mathbf{g}}{\partial \mathbf{u}} \cdot \delta\bar{\mathbf{u}} \, dS_T. \tag{97}$$

Introducing the adjoint structure, satisfying the boundary conditions

$$\mathbf{T}^{a0} = \frac{\partial \mathbf{g}}{\partial \mathbf{u}} \quad \text{on } S_T, \quad \mathbf{u}^{a0} = -\frac{\partial \mathbf{f}}{\partial \mathbf{T}} \quad \text{on } S_u, \tag{98}$$

and

$$\mathbf{f}_1{}^a = \frac{\partial h_1}{\partial \mathbf{u}} \quad \text{within } V_1, \quad \mathbf{f}_2{}^a = \frac{\partial h_2}{\partial \mathbf{u}} \quad \text{within } V_2, \tag{99}$$

and with the initial strain fields

$$\boldsymbol{\epsilon}_1{}^i = \frac{\partial \Psi_1}{\partial \boldsymbol{\sigma}_1} \quad \text{within } V_1, \quad \boldsymbol{\epsilon}_2{}^i = \frac{\partial \Psi_2}{\partial \boldsymbol{\sigma}_2} \quad \text{within } V_2, \tag{100}$$

the displacement field $\mathbf{u}^a$ and the stress field $\boldsymbol{\sigma}^r$ of this structure satisfy the conditions

$$[\![\mathbf{u}^a]\!] = 0, \quad [\![\mathbf{T}^r]\!] = [\![\boldsymbol{\sigma}^r]\!] \cdot \mathbf{n} = 0 \quad \text{on } S_c. \tag{101}$$

Following similarly as previously, we can write

$$\begin{aligned}
\int \frac{\partial \Psi_1}{\partial \boldsymbol{\sigma}_1} \cdot \delta\bar{\boldsymbol{\sigma}}_1 \, dV_1 + \int \frac{\partial \Psi_2}{\partial \boldsymbol{\sigma}_2} \cdot \delta\bar{\boldsymbol{\sigma}}_2 \, dV_2 &= \int \boldsymbol{\epsilon}_1^{\,i} \cdot \delta\bar{\boldsymbol{\sigma}}_1 \, dV_1 + \int \boldsymbol{\epsilon}_2^{\,i} \cdot \delta\bar{\boldsymbol{\sigma}}_2 \, dV_2 \\
&= \int \boldsymbol{\epsilon}_1^{\,a} \cdot \delta\bar{\boldsymbol{\sigma}}_1 \, dV_1 + \int \boldsymbol{\epsilon}_2^{\,a} \cdot \delta\bar{\boldsymbol{\sigma}}_1 \, dV_2 - \int \boldsymbol{\sigma}_1^{\,r} \cdot \delta\bar{\boldsymbol{\epsilon}}_2 \, dV_1 - \int \boldsymbol{\sigma}_2^{\,r} \cdot \delta\bar{\boldsymbol{\epsilon}}_2 \, dV_2 \\
&= \int \delta\mathbf{T} \cdot \mathbf{u}^{a0} \, dS_u - \int \mathbf{T}^{a0} \cdot \delta\bar{\mathbf{u}} \, dS_T + \int (\mathbf{T}^r \cdot [\![\mathbf{u}_{,n}]\!] - [\![\sigma_{kl}]\!]\epsilon^a_{kl} \\
&\quad + [\![\mathbf{f}]\!] \cdot \mathbf{u}^a)\delta\varphi_n \, dS_c - \int \mathbf{f}_1^{\,a} \cdot \delta\bar{\mathbf{u}} \, dV_1 - \int \mathbf{f}_2^{\,a} \cdot \delta\bar{\mathbf{u}} \, dV_2,
\end{aligned} \tag{102}$$

where the virtual strain equation (37) and stress equation (42) were used. The variation of G_1 can now be expressed as follows

$$\delta G_1 = \int ([\![\Psi]\!] + [\![h]\!] + \mathbf{T}^r \cdot [\![\mathbf{u}_{,n}]\!] - [\![\sigma_{kl}]\!]\epsilon^a_{kl} + [\![\mathbf{f}]\!] \cdot \mathbf{u}^a)\delta\varphi_n \, dS_c, \tag{103}$$

provided $\delta\varphi_n = 0$ on the curve Γ of intersection of the interface S_c with the exterior boundary.

Considering alternatively the functional

$$\begin{aligned}
G_2 = \int \Phi_1(\boldsymbol{\epsilon}_1) \, dV_1 + \int \Phi_2(\boldsymbol{\epsilon}_2) \, dV_2 + \int h_1(\mathbf{u}) \, dV_1 \\
+ \int h_2(\mathbf{u}) \, dV_2 + \int g(\mathbf{u}) \, dS_T + \int f(\mathbf{T}) \, dS_u,
\end{aligned} \tag{104}$$

its variation is expressed similarly to (103), that is

$$\delta G_2 = \int ([\![\Phi]\!] + [\![h]\!] + \mathbf{T}^r \cdot [\![\mathbf{u}_{,n}]\!] - [\![\sigma_{kl}]\!]\epsilon^a_{kl} + [\![\mathbf{f}]\!] \cdot \mathbf{u}^a)\delta\varphi_n \, dS_c. \tag{105}$$

Here $\mathbf{T}^r$ is the continuous contact traction at the interface S_c of the adjoint structure whereas ϵ^a_{kl} are the "internal" components of the strain field $\boldsymbol{\epsilon}^a$ of this structure. The associated quantities of the primary structure are the discontinuities in displacement gradient $\mathbf{u}_{,n}$ and in "internal" stress components σ_{kl}.

It can easily be shown that when G_1 and G_2 coincide with the complementary and potential energies, the derived expressions for variations of G_1 and G_2 coincide with those derived in the previous section for variations of Π_σ and Π_u.

4.2 *Translation and rotation of the boundary*

The general expressions for variations of G_1 and G_2 can now be particularized to the case of translation and rotation of the boundary.

When $\delta\varphi_k = \delta a_k = \text{const.}$ that is when the translation occurs, the general expression (83) takes the form

$$\delta G_1 = \int \delta\mathbf{T}^0 \cdot \mathbf{u}^a \, dS_T + \int [(\Psi + h)n_k + (\sigma^r_{ij}u_{i,k} - \sigma_{ij,k}u^a_i)n_j] \, dS_T \delta a_k, \tag{106}$$

and similar expression for the variation of G_2, namely

$$\delta G_2 = \int \delta \mathbf{T}^0 \cdot \mathbf{u}^a \, dS_T + \int [(\Phi + h)n_k + (\sigma_{ij}^r u_{i,k} - \sigma_{ij,k} u_i^a)n_j] \, dS_T \delta a_k. \tag{107}$$

When the variation $\delta \mathbf{T}^0$ associated with boundary translation vanishes, and G_1 coincides with the complementary energy, then $\Psi = W$, $\mathbf{T}^{a_0} = \boldsymbol{\sigma}^r \cdot \mathbf{n} = 0$, $\mathbf{u}^a = \mathbf{u}$. Moreover, setting $h = g = 0$, from (106), we obtain

$$\delta G_1 = \delta \Pi_\sigma = \int (W n_k - \sigma_{ij,k} u_i n_j) \, dS_T \delta a_k, \tag{108}$$

that is the relation (65). When on the other hand G_2 coincides with the potential energy, then $\Phi = U$, $\mathbf{T}^{a_0} = \boldsymbol{\sigma}^r \cdot \mathbf{n} = -\mathbf{T}^0$, $\mathbf{u}^a = 0$, and setting $h = f = 0$, from (107), we obtain

$$\delta G_2 = \delta \Pi_u = \int (U n_k - \mathbf{T}^0 \cdot \mathbf{u}_{,k}) \, dS_T \delta a_k, \tag{109}$$

that is the relation (64).

Similarly in the case of rotation of a closed boundary S_T, $\delta \varphi_k = e_{kjl} x_l \delta \omega_j$, the respective expressions for δG_1 and δG_2 are

$$\delta G_1 = \int \delta \mathbf{T}^0 \cdot \mathbf{u}^a \, dS_T + \int e_{kjl} \{[(\Psi + h)n_k + (\sigma_{ip}^r u_{i,k} - \sigma_{ip,k} u_i^a)n_p]x_l + \sigma_{il} u_i^a n_k\} \, dS_T \delta \omega_j, \tag{110}$$

and

$$\delta G_2 = \int \delta \mathbf{T}^0 \cdot \mathbf{u}^a \, dS_T + \int e_{kjl} \{[(\Phi + h)n_k + (\sigma_{ip}^r u_{i,k} - \sigma_{ip,k} u_i^a)n_p]x_l + \sigma_{il} u_i^a n_k\} \, dS_T \delta \omega_j. \tag{111}$$

When the variation $\delta \mathbf{T}^0$ associated with boundary rotation is defined by eqn (67), again the formula (69) is obtained when $\Psi = W$, $\mathbf{T}^{a_0} = 0$, $\mathbf{u}^a = \mathbf{u}$ and $h = g = 0$.

However, considering the expressions (106), (107) or (110), (111), a new class of conservation laws is generated. Consider, for example, the case of translation. Setting $\delta \mathbf{T}^0 = 0$, $h = f = 0$, and considering integral on any closed surface S within the body, from (106), we obtain for a homogeneous body (cf. Appendix B)

$$Z_T = \int (\Psi \delta_{kj} + \sigma_{ij}^r u_{i,k} - \sigma_{ij,k} u_i^a) n_j \, dS_T = 0. \tag{112}$$

The similar expression can be obtained from (107). These new conservation laws obviously generalize those derived in[8, 9, 11]. Their proof and application will be discussed in a separate paper.

5. STATIONARITY CONDITIONS IN OPTIMAL SHAPE DESIGN

In order to illustrate the applicability of derived expressions for first variations of functionals G_1 and G_2, let us discuss the optimal shape design. Assume the cost of the structure proportional to the material volume, thus

$$C = c \int dV, \quad \delta C = c \int n_k \delta \varphi_k \, dS, \tag{113}$$

where c is a constant parameter. The problem is then reduced to minimizing or maximizing the objective functional G with a specified upper bound on the structure cost, thus

$$\text{min. or max. of } G \quad \text{subject to } C \leqslant C_0. \tag{114}$$

Introducing the functional

$$G' = G + \lambda(C - C_0), \tag{115}$$

where λ is the Langrange multiplier, the condition of stationarity of G' is expressed as follows

$$\delta G' = \delta G + \lambda \delta C + \delta\lambda(C - C_0) = 0, \tag{116}$$

and

$$\delta G = -\lambda \delta C, \quad \delta\lambda(C - C_0) = 0. \tag{117}$$

The second equality requires either $C = C_0$ or $\delta\lambda = 0$. The first condition can be expressed explicitly by using the respective expression for variation of G and (113).

In particular, when $G = G_1$ and G_1 is defined by (74), the optimality condition (116) is expressed as follows

$$\int \delta \mathbf{T}^0 \cdot \mathbf{u}^a \, \mathrm{d}S_T + \int \{[\Psi + h + g_{,n} + (\mathbf{T}^0 \cdot \mathbf{u}^a)_{,n} - 2(g + \mathbf{T}^0 \cdot \mathbf{u}^a)H - \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^a$$
$$+ \mathbf{f} \cdot \mathbf{u}^a]n_k - \mathbf{T}^0_{,k} \cdot \mathbf{u}^a\} \delta\varphi_k \, \mathrm{d}S_T = -\lambda c \int n_k \delta\varphi_k \, \mathrm{d}S_T, \tag{118}$$

where the expression (84) for δG_1 was used. When, in particular, G_1 coincides with the complementary energy Π_σ, then $\Psi(\boldsymbol{\sigma}) = W(\boldsymbol{\sigma})$ and in view of (55), we have

$$\int \{[W + (\mathbf{T}^0 \cdot \mathbf{u})_{,n} - 2\mathbf{T}^0 \cdot \mathbf{u}H - \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} + \mathbf{f} \cdot \mathbf{u}]n_k - \mathbf{T}^0_{,k} \cdot \mathbf{u}\} \delta\varphi_k \, \mathrm{d}S_T$$
$$+ \int \delta \mathbf{T}^0 \cdot \mathbf{u} \, \mathrm{d}S_T = -\lambda c \int n_k \delta\varphi_k \, \mathrm{d}S_T. \tag{119}$$

In the case of free boundary variation ($\mathbf{T}^0 = 0$) in the absence of body forces and with $h(\mathbf{u}) = 0$ within V, $g(\mathbf{u}) = 0$ on S_0, from (85) it follows that

$$\int (\Psi - \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^a) \delta\varphi_n \, \mathrm{d}S_0 = -\lambda c \int \delta\varphi_n \, \mathrm{d}S_0, \tag{120}$$

and the local optimality condition is

$$\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^a - \Psi = \lambda c = \text{const. on } S_0. \tag{121}$$

Finally, when only supported boundary S_u is subject to variation, the optimality conditions for the objective functional (74), in view of (86), are expressed as follows

$$\int \{[\Psi + h + f_{,n} + (\mathbf{T} \cdot \mathbf{u}^a)_{,n} - 2(f + \mathbf{T} \cdot \mathbf{u}^{a0})H - \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^a + \mathbf{f} \cdot \mathbf{u}^{a0}]n_k$$
$$+ \mathbf{T}^a \cdot \mathbf{u}_{,k}\} \delta\varphi_k \, \mathrm{d}S_u = -\lambda c \int n_k \delta\varphi_k \, \mathrm{d}S_u. \tag{122}$$

Similarly, the optimality conditions can easily be stated for the objective functional G_2 expressed by (87). In fact, using (93) and (113), the optimality conditions for the loaded boundary variation are

$$\int \{[\Phi + h + g_{,n} + (\mathbf{T}^0 \cdot \mathbf{u}^a)_{,n} - 2(g + \mathbf{T}^0 \cdot \mathbf{u}^a)H - \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^a + \mathbf{f} \cdot \mathbf{u}^a]n_k - \mathbf{T}^0_{,k} \cdot \mathbf{u}^a\}\delta\varphi_k \, dS_T$$

$$+ \int \delta \mathbf{T}^0 \cdot \mathbf{u}^a \, dS_T = -\lambda c \int n_k \delta\varphi_k \, dS_T, \tag{123}$$

and for the variation of free boundary, in the absence of body forces and with $h(\mathbf{u}) = 0$ within V, $g(\mathbf{u}) = 0$ on S_0, the local optimality condition is

$$\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^a - \Phi = \lambda c = \text{const. on } S_0. \tag{124}$$

In particular, for the global compliance design, when G_2 coincides with the global potential energy Π_u and thus $\Phi(\epsilon) = U(\epsilon)$, the condition (124) becomes

$$U = -\lambda c = \text{const. on } S_0. \tag{125}$$

The sufficient optimality conditions for that case of design were derived in[2, 3]. They require the specific strain energy to be constant on S_0 and a decreasing function along the exterior normal to the boundary.

When only variation of supported boundary is considered, the optimality conditions for the objective functional G_2 (87), in view of (95), become

$$\int \{[\Phi + h + f_{,n} + (\mathbf{T} \cdot \mathbf{u}^a)_{,n} - 2(f + \mathbf{T} \cdot \mathbf{u}^{a0})H - \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^a + \mathbf{f} \cdot \mathbf{u}^{a0}]n_k$$

$$+ \mathbf{T}^a \cdot \mathbf{u}_{,k}\}\delta\varphi_k \, dS_u = -\lambda c \int n_k \delta\varphi_k \, dS_u. \tag{126}$$

For the case of mean compliance design, when G_2 coincides with the global potential energy Π_u, the local optimality condition for rigid support on S_u follows directly from (126), namely

$$U - \mathbf{T} \cdot \mathbf{u}_{,n} = -\lambda c = \text{const. on } S_u. \tag{127}$$

For the variation of the interface S_c, in view of (103) and (105), the optimality conditions are

$$\begin{gathered} [\![\Psi]\!] + [\![h]\!] + \mathbf{T}^r \cdot [\![\mathbf{u}_{,n}]\!] - [\![\sigma_{kl}]\!]\epsilon^a_{kl} + [\![\mathbf{f}]\!] \cdot \mathbf{u}^a = -\lambda(c_1 - c_2) = \text{const. on } S_c, \\ \delta\lambda(C - C_0) = 0, \end{gathered} \tag{128}$$

and

$$\begin{gathered} [\![\Phi]\!] + [\![h]\!] + \mathbf{T}^r \cdot [\![\mathbf{u}_{,n}]\!] - [\![\sigma_{kl}]\!]\epsilon^a_{kl} + [\![\mathbf{f}]\!] \cdot \mathbf{u}^a = -\lambda(c_1 - c_2) = \text{const. on } S_c, \\ \delta\lambda(C - C_0) = 0, \end{gathered} \tag{129}$$

where c_1 and c_2 denote specific costs of the portions V_1 and V_2, so that

$$\delta C = (c_1 - c_2) \int n_k \delta\varphi_k \, dS_c. \tag{130}$$

For the case of mean compliance design, when G_1 coincides with the complementary energy Π_σ and G_2—with potential energy Π_u, the conditions (128) and (129) become

$$[\![\sigma_{kl}]\!]\epsilon_{kl} - [\![W]\!] - [\![\mathbf{f}]\!]\cdot\mathbf{u} = \lambda(c_1 - c_2) = \text{const. on } S_c, \tag{131}$$

and

$$[\![U]\!] - [\![\mathbf{f}]\!]\cdot\mathbf{u} - \mathbf{T}^c\cdot[\![\mathbf{u}_{,n}]\!] = -\lambda(c_1 - c_2) = \text{const. on } S_c. \tag{132}$$

The equivalence of (131) and (132) follows from the equality

$$[\![W]\!] + [\![U]\!] = [\![\sigma\cdot\epsilon]\!] = [\![\sigma_{kl}]\!]\epsilon_{kl} + \sigma_{in}[\![\epsilon_{in}]\!], \tag{133}$$

valid for $k, l = 1, 2$, $i = 1, 2, 3$ and $n = 3$, where k, l, n is the local coordinate system with k, l-axes lying in the plane tangential to S_c.

The application of the optimality condition (131) in optimal design of stepped plates was presented in[4].

5. EXAMPLES

In this Section, two simple examples of application of the derived optimality conditions are presented. Further examples can be found in previous works[3–5].

Example 1. *Prismatic bar under torsion and bending*

Consider a prismatic bar of elliptic cross-section, subjected to combined torsion and bending by the moments M_t and M_b, Fig. 3(a). We shall look for an optimal cross sectional shape within the class of elliptic shapes of specified cross-section area A_0 and for the stress constraint

$$\sigma_e = [\sigma_{33}^2 + 3(\sigma_{13}^2 + \sigma_{23}^2)]^{1/2} \leqslant \sigma_0, \tag{134}$$

where σ_{13}, σ_{23} and σ_{33} are the non-vanishing shear and normal stress components within the bar, refered to the coordinate system (x_1, x_2, x_3). These components are expressed as follows in terms of the bending and torsional moments

$$\sigma_{33} = \frac{4M_b}{\pi ab^3}x_2, \quad \sigma_{13} = -\frac{2M_t}{\pi ab^3}x_2, \quad \sigma_{23} = \frac{2M_t}{\pi a^3 b}x_1. \tag{135}$$

Instead of the condition (134), we shall minimize the functional

$$G_1 = \int\left(\frac{\sigma_e}{\sigma_0}\right)^m \mathrm{d}A = \frac{[\sigma_{33}^2 + 3(\sigma_{13}^2 + \sigma_{23}^2)]^{m/2}}{\sigma_0^m}\mathrm{d}A \rightarrow \text{min.}, \tag{136}$$

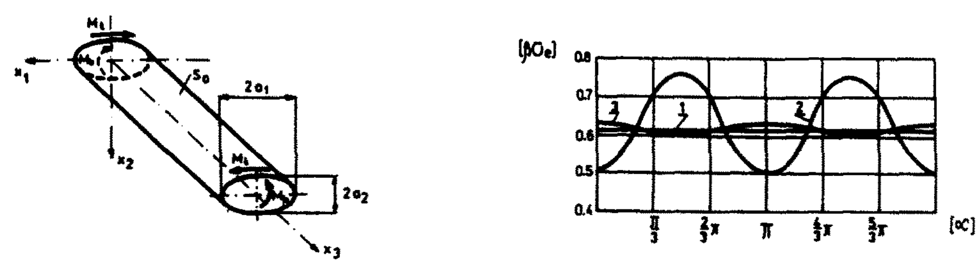

Fig. 3. Prismatic bar under torsion and bending; (a) Bar subjected to combined torsion and bending by the moments M_t and M_b, (b) Distribution of the effective stress σ_e along the cross-sectional perimeter ($M_t/M_b = 1$, $\nu = 0.3$).

subject to the constraint

$$\pi a_1 a_2 = A_0 = \text{const.} \tag{137}$$

Let us note that for $m \to \infty$ the functional G_1 represents the local effective stress defined by (134). The shape variation now depends on two parameters a_1 and a_2, and the stationarity condition (120) now takes the form

$$\int (\Psi - \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^a)\delta\varphi_n \, \mathrm{d}S_0 = \lambda c \int \delta\varphi_n \, \mathrm{d}S_0, \tag{138}$$

where

$$\Psi = \left(\frac{\sigma_e}{\sigma_0}\right)^m, \tag{139}$$

and the initial strains within the adjoint system are

$$\epsilon_{33}^i = \frac{\partial \Psi}{\partial \sigma_{33}} = \frac{m}{\sigma_0^m}\sigma_e^{m-2}\sigma_{33}, \quad \epsilon_{13}^i = \frac{\partial \Psi}{\partial \sigma_{13}} = \frac{3m}{\sigma_0^m}\sigma_e^{m-2}\sigma_{13},$$

$$\epsilon_{23}^i = \frac{\partial \Psi}{\partial \sigma_{23}} = \frac{3m}{\sigma_0^m}\sigma_e^{m-2}\sigma_{23}, \tag{140}$$

$$\boldsymbol{\epsilon}^r = 0, \quad \boldsymbol{\epsilon}^a = \boldsymbol{\epsilon}^i.$$

The variation of the cross-sectional shape occurs due to variation of its semiaxes a_1 and a_2, thus

$$\delta\varphi_n = n_k \frac{\partial \varphi_k}{\partial a_l}\delta a_l. \tag{141}$$

In view of (140) and (141), the stationarity condition (138) takes the form

$$(m-1)\int \left(\frac{\sigma_e}{\sigma_0}\right)^m \frac{x_1}{a_1} n_1 \, \mathrm{d}S_0 = \lambda c \int \frac{x_1}{a_1} n_1 \, \mathrm{d}S_0,$$

$$(m-1)\int \left(\frac{\sigma_e}{\sigma_0}\right)^m \frac{x_2}{a_2} n_2 \, \mathrm{d}S_0 = \lambda c \int \frac{x_2}{a_2} n_2 \, \mathrm{d}S_0, \tag{142}$$

$$a_1 a_2 = \frac{A_0}{\pi} = \text{const.}$$

The optimal values of a_1 and a_2 are calculated from (142)

$$a_2 = \sqrt{\frac{A_0}{\pi}} \sqrt[4]{\frac{4}{3}\left(\frac{M_b}{M_t}\right)^2 + 1}, \quad a_1 = \frac{A_0}{\pi a_2}. \tag{143}$$

Figure 3(b) presents the distribution of the effective stress along the cross-section perimeter (curve 1). It is seen that σ_e is constant for the shape specified by (143).

Consider now the mean stiffness design for which the complementary energy is minimized

$$\Pi_\sigma(\mathbf{a}, \boldsymbol{\sigma}) = \int W(\boldsymbol{\sigma}) \, \mathrm{d}A \to \min_{\mathbf{a}}. \tag{144}$$

The optimal values of a_1 and a_2 now are

$$a_2 = \sqrt{\frac{A_0}{\pi}}\sqrt[4]{\frac{2}{1+\nu}\left(\frac{M_b}{M_t}\right)^2 + 1}, \quad a_1 = \frac{A_0}{\pi a_2}. \tag{145}$$

Curve 3 in Fig. 3(b) represents the distribution of σ_e along the boundary for the design specified by (145). It is seen that σ_e does vary significantly. On the other, for a circular cross section of the same area the variation of σ_e is quite considerable, (curve 2 in Fig. 3b).

Example 2. *Design of annular disk*

Consider a circular disk of radius r_e with a central of radius r_i, loaded uniformly by pressures p_e and p_i, Fig. 4(a). Consider the mean compliance design for which both radii r_i and r_e are to be determined such that the complementary energy

$$\Pi_\sigma = \frac{1}{2E}\int_{r_i}^{r_e} (\sigma_r^2 - 2\nu\sigma_r\sigma_t + \sigma_t^2) r \, dr \tag{146}$$

attains a minimum subject to the condition of constant material cost

$$C = c\pi(r_e^2 - r_i^2) = C_0, \tag{147}$$

where σ_r and σ_t are radial and circumferential stresses, and E, ν denote the elastic constants. The optimality conditions in the case of pressure loaded boundary follows from (57) and have the form[5]

$$\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} - W - \operatorname{div}(p\mathbf{u}) = \lambda c, \quad C = C_0, \tag{148}$$

which in our case become

$$(\sigma_t + p_i)^2 - 2(1-\nu)p_i^2 = 2\lambda c E \quad \text{for } r = r_i,$$
$$(\sigma_t + p_e)^2 - 2(1-\nu)p_e^2 = 2\lambda c E \quad \text{for } r = r_e. \tag{149}$$

The stress state within the disk is expressed as follows

$$\sigma_r = \frac{A}{r^2} + B, \quad \sigma_t = -\frac{A}{r^2} + B, \tag{150}$$

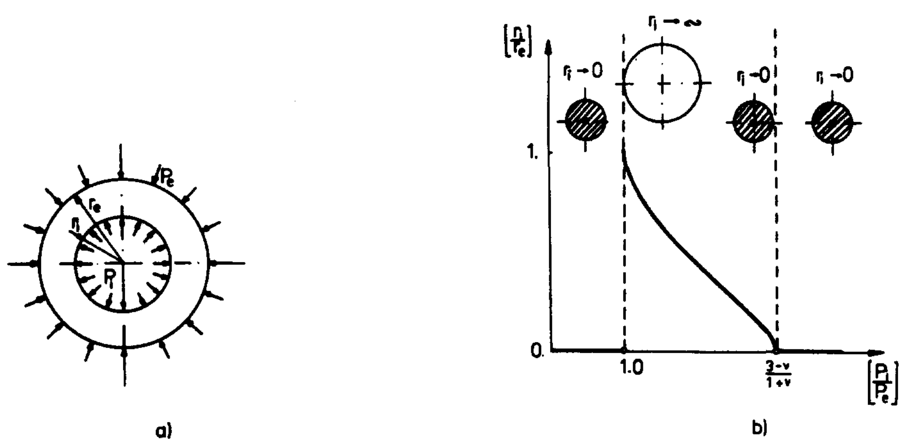

Fig. 4. Design of annular disk; (a) Disk loaded uniformly by pressures p_e and p_i, (b) Variation of r_i/r_e in function of p_i/p_e.

where

$$A = \frac{r_i^2 r_e^2}{r_e^2 - r_i^2}(p_e - p_i), \quad B = \frac{p_i r_i^2 - p_e r_e^2}{r_e^2 - r_i^2}. \tag{151}$$

Substituting (150) into (149), and solving for r_i, r_e, we obtain

$$r_i = \frac{1}{2}\sqrt{\frac{C_0}{\pi c}\frac{(3-\nu)p_e - (1+\nu)p_i}{p_i - p_e}}, \quad r_e = \frac{1}{2}\sqrt{\frac{C_0}{\pi c}\frac{(3-\nu)p_i - (1+\nu)p_e}{p_i - p_e}}, \tag{152}$$

valid for

$$1 < \frac{p_i}{p_e} < \frac{3-\nu}{1+\nu}. \tag{153}$$

Figure 4(b) presents the variation of r_i/r_e in function of p_i/p_e. It is seen that for $p_i/p_e < 1$ the optimal solution corresponds to the vanishing hole, whereas for p_i/p_e varying within the range corresponding to the inequality (153) the disk is gradually transformed from a thin ring into a circular disk without the hole.

6. CONCLUDING REMARKS

The present work is supplementary to Ref. [1] and provides a systematic variational approach to sensitivity analysis and optimal design of a structure with shape variations of its boundaries. It summarizes and extends previous results obtained in[1–5]. The analysis is limited to linearly elastic structures for which the concept of an adjoint structure can easily be applied in order to derive the expression for first variation of any volume or surface integral.

Besides optimal design or identification problems the present approach can also be applied in study of fracture problems, metallurgical transformations, grain boundary movements or growth process of biological materials. In these cases the transformation field $\varphi(x)$ is specified by a growth or transformation rule, relating the rate of growth to mechanical or chemical state parameters. This new and unexplored areas of structure transformation will be discussed in more detail in future papers from the point of view of sensitivity analysis and optimality conditions.

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APPENDIX A

Material derivatives on a surface element

In order to make our paper sensibly self-contained, we present briefly the derivation of material rates of normal and tangential tractions and forces acting on the surface element undergoing transformation. The variations used in the paper can thus be identified with material rates. A systematic discussion of material rates of surface data was recently presented by Hill[12].

(1) *Material rate of a normal unit vector to the surface.* Consider a material surface element in motion associated with transformation

$$f^0(\mathbf{x}, t) = 0. \tag{A1}$$

Then, the material derivative of (A1) is

$$\dot{f}^0 = \frac{\partial f^0}{\partial t} + \mathbf{f}\cdot\mathbf{v} = 0 \tag{A2}$$

where $\dot{\varphi} = \mathbf{v}$ denotes the transformation velocity vector on the surface, and $\mathbf{f} = \partial f^0/\partial \mathbf{x}$ is the gradient of surface (A1). The material derivative of $\mathbf{f}$ is expressed similarly

$$\dot{f}_i = \left(\frac{\partial f^0}{\partial x_i}\right)^{\cdot} = \frac{\partial^2 f^0}{\partial x_i \partial t} + \frac{\partial^2 f^0}{\partial x_i \partial x_k} v_k. \tag{A3}$$

However, from (A2) it follows that

$$\frac{\partial^2 f^0}{\partial x_i \partial t} + \frac{\partial^2 f^0}{\partial x_i \partial x_k} v_k + \frac{\partial f^0}{\partial x_k}\frac{\partial v_k}{\partial x_i} = 0. \tag{A4}$$

In view of (A3) and (A4), we have

$$\dot{f}_i = -\frac{\partial f^0}{\partial x_k}\frac{\partial v_k}{\partial x_i} = -f_k v_{k,i}$$

or

$$\dot{\mathbf{f}} = -\mathbf{L}^T\cdot\mathbf{f} \tag{A5}$$

where $L_{ij} = v_{i,j} = \dot{\varphi}_{i,j}$ and $L^T_{ji} = L_{ij}$ is transpose velocity gradient matrix.

The unit normal vector to the surface (A1) is expressed as follows

$$\mathbf{n} = \frac{\mathbf{f}}{(\mathbf{f}\cdot\mathbf{f})^{1/2}} = \frac{\mathbf{f}}{f} \tag{A6}$$

and in view of (A5) its rate equals

$$\dot{\mathbf{n}} = \frac{1}{f}\dot{\mathbf{f}} - \frac{\dot{f}}{f^2}\mathbf{f} = -\mathbf{L}^T\cdot\mathbf{n} + (\mathbf{n}\cdot\mathbf{L}^T\cdot\mathbf{n})\mathbf{n} \tag{A7}$$

and finally

$$\dot{\mathbf{n}} = (\mathbf{n}\cdot\mathbf{L}\cdot\mathbf{n})\mathbf{n} - \mathbf{L}^T\cdot\mathbf{n} = \mathbf{n}(\mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n}) + \omega\cdot\mathbf{n} - \mathbf{D}\cdot\mathbf{n} \tag{A8}$$

where

$$2\mathbf{D} = \mathbf{L} + \mathbf{L}^T, \quad 2\omega = \mathbf{L} - \mathbf{L}^T \tag{A9}$$

are the associated strain and rotation rates.

Equation (A8) can be written in an alternative form

$$\dot{n}_i = (n_i n_l - \delta_{il}) n_k v_{k,l}. \tag{A10}$$

(2) *Material rate of a tangential unit vector on the surface.* Consider now any curve Γ lying on the surface (A1), given in the form

$$\mathbf{x} = \mathbf{x}(\tau) \tag{A11}$$

where τ is a parameter specifying Γ. The vector tangential to this curve can be expressed as follows

$$\mathbf{x}_{,\tau} = \frac{d\mathbf{x}}{d\tau} \tag{A12}$$

and its rate equals

$$\dot{x}_{i,\tau} = v_{i,j} x_{j,\tau}$$

or

$$\dot{\mathbf{x}}_{,\tau} = \mathbf{L} \cdot \mathbf{x}_{,\tau} \tag{A13}$$

where, as previously, $L_{ij} = v_{i,j} = \dot{\varphi}_{i,j}$.

The unit tangential vector to the curve (A11) is now expressed as follows

$$\mathbf{t} = \frac{\mathbf{x}_{,\tau}}{(\mathbf{x}_{,\tau} \cdot \mathbf{x}_{,\tau})^{1/2}} = \frac{\mathbf{x}_{,\tau}}{x_{,\tau}} \tag{A14}$$

and in view of (A13), its rate equals

$$\dot{\mathbf{t}} = \frac{1}{x_{,\tau}} \dot{\mathbf{x}}_{,\tau} - \frac{\dot{x}_{,\tau}}{x_{,\tau}^2} \mathbf{x}_{,\tau} = \mathbf{L} \cdot \mathbf{t} - (\mathbf{t} \cdot \mathbf{L} \cdot \mathbf{t})\mathbf{t}. \tag{A15}$$

Thus, finally we can write

$$\dot{\mathbf{t}} = \mathbf{L} \cdot \mathbf{t} - (\mathbf{t} \cdot \mathbf{L} \cdot \mathbf{t})\mathbf{t} = \mathbf{D} \cdot \mathbf{t} + \omega \cdot \mathbf{t} - \mathbf{t}(\mathbf{t} \cdot \mathbf{D} \cdot \mathbf{t}) \tag{A16}$$

where $\mathbf{D}$ and ω are defined by eqn (A9). The alternative form of (A16) is

$$\dot{t}_i = (\delta_{ik} - t_i t_k) v_{k,j} t_j. \tag{A17}$$

Since eqn (A11) describes any curve lying on the surface (A1), then relation (A16) or (A17) defines the material rate of any unit vector $\mathbf{t}$ being tangential to the surface (A1).

(3) *Material rate of the surface element area.* For any material line element $\mathbf{dx}$ on the surface S (A1) there is

$$\dot{\mathbf{d}} \cdot \mathbf{x} = \mathbf{L} \cdot \mathbf{dx} \quad \text{or} \quad \dot{dx}_i = L_{ij} dx_j \tag{A18}$$

Consider two infinitesimal material elements $\mathbf{dx}^1$ and $\mathbf{dx}^2$. The vector of surface element area is then, see Fig. A1

$$\mathbf{dS} = \mathbf{dx}^1 \chi\, \mathbf{dx}^2 = \mathbf{n}\, dS \quad \text{or} \quad dS_i = e_{ijk} dx_j^1 dx_k^2 \tag{A19}$$

where χ denotes the vector product and $\mathbf{n}$ is the unit normal vector to S; dS is the area of surface element and e_{ijk} denotes the permutation symbol.

Equation (A19) can be now expressed as follows

$$dS_i = \frac{1}{2} e_{ijk} (dx_j^1 dx_k^2 - dx_k^1 dx_j^2) \tag{A20}$$

and in view of (A18) its material rate is

$$\dot{dS}_i = \frac{1}{2} e_{ijk} L_{jm} (dx_m^1 dx_k^2 - dx_k^1 dx_m^2) + \frac{1}{2} e_{ijk} L_{km} (dx_j^1 dx_m^2 - dx_m^1 dx_j^2). \tag{A21}$$

Multiplying now eqn (A13) be e_{mni} it is obtained

$$e_{mni} dS_i = dx_m^1 dx_n^2 - dx_n^1 dx_m^2. \tag{A22}$$

Using this relation in (A21), the material rate of surface element vector can be expressed as follows

$$\dot{dS}_i = L_{kk} dS_i - L_{ji} dS_j$$

or

$$\dot{\mathbf{d}}\mathbf{S} = (\operatorname{div} \mathbf{v})\mathbf{dS} - \mathbf{L}^T \cdot \mathbf{dS} \tag{A23}$$

that can be written alternatively as

$$\dot{dS}_i = (n_i v_{k,k} - n_j v_{j,i}) dS \tag{A24}$$

Calculating now the material derivative of the first equality of (A19), we can write

$$\dot{\mathbf{d}}\mathbf{S} = \dot{dS}\,\mathbf{n} + dS\,\dot{\mathbf{n}} \tag{A25}$$

in view of (A8), (A19) and (A23), we obtain

$$(\operatorname{div} \mathbf{v})\, dS\, \mathbf{n} = \dot{dS}\ \mathbf{n} + (\mathbf{n} \cdot \mathbf{L} \cdot \mathbf{n})\, dS\, \mathbf{n}. \tag{A26}$$

Thus, the material rate of element area equals

$$\dot{dS} = (\operatorname{div} \mathbf{v} - \mathbf{n} \cdot \mathbf{L} \cdot \mathbf{n})\, dS \tag{A27}$$

or alternatively

$$d\dot{S} = (v_{k,k} - n_l n_p v_{p,l})\, dS. \tag{A28}$$

(4) *Material rate of surface tractions*. Consider the surface traction being permanently normal to the loaded surface, for which intensity factor equals p. The normal stress vector at any point P of boundary surface can be expressed as follows

$$\boldsymbol{\sigma}_n = p\, \mathbf{n} \tag{A29}$$

and the normal force vector acting on surface element dS takes the form

$$\mathbf{R}_n = \boldsymbol{\sigma}_n\, dS = p\, \mathbf{n}\, dS = p\, \mathbf{dS}. \tag{A30}$$

To calculate the material rate of stress vector $\boldsymbol{\sigma}_n$, in view of (A29), we can write

$$\dot{\boldsymbol{\sigma}}_n = \dot{p}\, \mathbf{n} + p\, \dot{\mathbf{n}}. \tag{A31}$$

Using now (A8) and (A29) in (A31), the material rate of normal stress vector $\boldsymbol{\sigma}_n$ is expressed in the form

$$\dot{\boldsymbol{\sigma}}_n = (\mathbf{n} \cdot \mathbf{L} \cdot \mathbf{n})\boldsymbol{\sigma}_n - \mathbf{L}^T \cdot \boldsymbol{\sigma}_n + \frac{\dot{p}}{p}\boldsymbol{\sigma}_n \tag{A32}$$

or, in view of (A9), as

$$\dot{\boldsymbol{\sigma}}_n = (\mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n})\boldsymbol{\sigma}_n + \omega \cdot \boldsymbol{\sigma}_n - \mathbf{D} \cdot \boldsymbol{\sigma}_n + \frac{\dot{p}}{p}\boldsymbol{\sigma}_n. \tag{A33}$$

Similarly, it follows from (A30)

$$\dot{\mathbf{R}}_n = \dot{p}\, \mathbf{dS} + p\, \mathbf{d\dot{S}} \tag{A34}$$

and by using (A23), the material rate of normal force vector $\mathbf{R}_n$ is expressed in the form

$$\dot{\mathbf{R}}_n = (\operatorname{div} \mathbf{v})\mathbf{R}_n - \mathbf{L}^T \cdot \mathbf{R}_n + \frac{\dot{p}}{p}\mathbf{R}_n. \tag{A35}$$

In a Cartesian reference system, the relations (A33) and (A35) can be expressed in a matrix form

$$\{\dot{\sigma}_{n_i}\} = [a_{ij}(v_{k,l};\, n_p)]\{\sigma_{n_j}\} + \alpha\{\sigma_{n_i}\} \tag{A36}$$

and

$$\{\dot{R}_{n_i}\} = [A_{ij}(v_{k,l})]\{R_{n_j}\} + \alpha\{R_{n_i}\}. \tag{A37}$$

A quite different kind of loading occurs in the case of tangential follower force. Here, the local traction is tangential to an embedded fibre element of the surface. In this case, the tangential stress vector at any point P of the loaded surface can be expressed in the form

$$\boldsymbol{\sigma}_t = q\, \mathbf{t} \tag{A38}$$

where $\mathbf{t}$ denotes unit tangential vector on S and q is intensity factor of tangential force. In view of (A16), the material rate of (A38) is expressed as follows

$$\dot{\boldsymbol{\sigma}}_t = \mathbf{L} \cdot \boldsymbol{\sigma}_t - (\mathbf{t} \cdot \mathbf{L} \cdot \mathbf{t})\boldsymbol{\sigma}_t + \frac{\dot{q}}{q}\boldsymbol{\sigma}_t \tag{A39}$$

or, in view of (A9), it takes the form

$$\dot{\boldsymbol{\sigma}}_t = \mathbf{D} \cdot \boldsymbol{\sigma}_t + \omega \cdot \boldsymbol{\sigma}_t - (\mathbf{t} \cdot \mathbf{D} \cdot \mathbf{t})\boldsymbol{\sigma}_t + \frac{\dot{q}}{q}\boldsymbol{\sigma}_t. \tag{A40}$$

The tangential force vector is defined by

$$\mathbf{R}_t = q\mathbf{t}\, dS = \boldsymbol{\sigma}_t dS \tag{A41}$$

and in view of (A16) and (A27), its material rate equals

$$\dot{\mathbf{R}}_t = \mathbf{L} \cdot \mathbf{R}_t - (\mathbf{t} \cdot \mathbf{L} \cdot \mathbf{t} + \mathbf{n} \cdot \mathbf{L} \cdot \mathbf{n})\mathbf{R}_t + (\operatorname{div} \mathbf{v})\mathbf{R}_t + \frac{\dot{q}}{q}\mathbf{R}_t. \tag{A42}$$

In the Cartesian system (x_1, x_2, x_3), these rates are expressed in the matrix form

$$\{\dot{\sigma}_{t_i}\} = [b_{ij}(v_{k,l};\, t_p)]\{\sigma_{t_j}\} + \beta\{\sigma_{t_i}\} \tag{A43}$$

and

$$\{\dot{R}_{,i}\} = [B_{ij}(v_{k,l}; t_p; n_r)]\{R_{,j}\} + \beta\{R_{,i}\}. \tag{A44}$$

Let us note, that for both of considered loadings their material rates are the sum of two parts—the first part being the controllable part independent of material response and the second one being the transformation-sensitive part depending linearly on the velocity gradient.

APPENDIX B

Proof of the conservation rule (112) for an arbitrary stress functional

Consider the integral (112), that is

$$Z_k = \int (\Psi(\sigma_{mn})\delta_{kj} + \sigma^r_{ij}u_{i,k} - \sigma_{ij,k}u^a_i)n_j dS \tag{B1}$$

specified on any closed surface S within the *homogeneous* body. Here $\Psi(\sigma_{mn})$ is an arbitrary scalar stress function possessing uniquely defined gradient $\partial\Psi/\partial\sigma_{mn}$. The adjoint body is specified by (76)–(79) with $g = f = h = O$. The stress field within the adjoint body is σ^r_{ij} whereas u^a_i is the displacement field. In view of (77) and (78) the initial strain field within the adjoint body is

$$\epsilon^{(i)}_{ij} = \frac{\partial\Psi}{\partial\sigma_{ij}}, \quad \epsilon^a_{ij} = \epsilon^r_{ij} + \epsilon^i_{ij} \tag{B2}$$

and σ^r_{ij} is the residual stress field induced by the initial strain field ϵ^i_{ij}.

Transform the surface integral (B1) into the volume integral

$$\begin{aligned} Z_k &= \int [\Psi_{,k} + (\sigma^r_{ij}u_{i,k})_{,j} - (\sigma_{ij,k}u^a_i)_{,j}]\,dV_s \\ &= \int \left[\frac{\partial\Psi}{\partial\sigma_{ij}}\sigma_{ij,k} + \sigma^r_{ij,j}u_{i,k} + \sigma^r_{ij}u_{i,kj} - (\sigma_{ij,j})_{,k}u^a_i - \sigma_{ij,k}u^a_{i,j}\right]dV_s \\ &= \int \left[\frac{\partial\psi}{\partial\sigma_{ij}}\sigma_{ij,k} + \sigma^r_{ij}\epsilon_{ij,k} - \sigma_{ij,k}\epsilon^a_{ij}\right]dV_s \end{aligned} \tag{B3}$$

where V_s is the volume of the domain enclosed by the surface S. In (B3) the second and the fourth terms vanish by virtue of equilibrium conditions for stress fields σ_{ij} and σ^r_{ij}, that is $\sigma_{ij,j} = \sigma^r_{ij,j} = 0$. in view of (B2), the expression (B3) is further retransformed as follows

$$Z_k = \int [\epsilon^i_{ij}\sigma_{ij,k} + \sigma^r_{ij}\epsilon_{ij,k} - \epsilon^r_{ij}\sigma_{ij,k} - \epsilon^i_{ij}\sigma_{ij,k}]\,dV_s = \int [\sigma^r_{ij}\epsilon_{ij,k} - \epsilon^r_{ij}\sigma_{ij,k}]\,dV_s. \tag{B4}$$

For a homogeneous body, there is

$$\begin{aligned} \sigma^r_{ij}\epsilon_{ij,k} &= D_{ijmn}\epsilon^r_{mn}\epsilon_{ij,k} \\ \sigma_{ij,k}\epsilon^r_{ij} &= (D_{ijmn}\epsilon_{mn})_{,k}\epsilon^r_{ij} = D_{ijmn}\epsilon_{mn,k}\epsilon^r_{ij} = D_{ijmn}\epsilon^r_{mn}\epsilon_{ij,k} \end{aligned} \tag{B5}$$

since the stiffness matrix does not vary with position $D_{ijmn,k} = 0$. Thus $\sigma^r_{ij}\epsilon_{ij,k} - \epsilon^r_{ij}\sigma_{ij,k} = 0$, and

$$Z_k = 0 \tag{B6}$$

for any closed surface S within the body considered.

Consider the stress functional

$$G_1 = \int \Psi(\sigma_{ij})\,dV. \tag{B7}$$

Since $\delta G_1 = Z_k\delta a_k$ for the translation of any closed surface, the stress functional G_1 preserves constant value during translation of the body.